Welschinger invariants of toric Del Pezzo surfaces with non-standard real structures

E. Shustin*

Abstract

The Welschinger invariants of real rational algebraic surfaces are natural analogues of the Gromov-Witten invariants, and they estimate from below the number of real rational curves passing through prescribed configurations of points. We establish a tropical formula for the Welschinger invariants of four toric Del Pezzo surfaces, equipped with a non-standard real structure. Such a formula for real toric Del Pezzo surfaces with a standard real structure (i.e., naturally compatible with the toric structure) was established by Mikhalkin and the author. As a consequence we prove that, for any real ample divisor D on a surfaces Σ under consideration, through any generic configuration of $c_1(\Sigma)D-1$ generic real points there passes a real rational curve belonging to the linear system $|D|$.

Introduction

The Welschinger invariants [19, 20] play a central role in the enumerative geometry of real rational curves on real rational surfaces, providing lower bounds for the number of real rational curves passing through generic, conjugation invariant configurations of points, whereas the number of respective complex curves (Gromov-Witten invariant) serves as an upper bound. Methods of the tropical enumerative geometry, developed in [13, 14, 17], allowed one to compute and estimate the Welschinger invariants for the real toric Del Pezzo surfaces, equipped with the standard real structure [7, 9, 18]: the plane $\mathbb{P}^2$, the plane $\mathbb{P}^2_k$ with blown up $k = 1$, 2, or 3 real (generic) points, and the quadric $(\mathbb{P}^1)^2$.

We notice that the available technique of the tropical enumerative geometry applies only to toric surfaces, and among them the Welschinger invariant is well-defined only for unndodal[1] Del Pezzo surfaces. So, the main goal of this paper is to compute Welschinger invariants for other real toric Del Pezzo surfaces using the tropical enumerative geometry.

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[1]Like in [7, 9] "unnodal" means the absence of $(-n)$-curves, $n \geq 2$.

Along the Comessatti's classification of real rational surfaces [1, 2] (see also [12]), besides the standard real toric Del Pezzo surfaces, there are five more types, having a non-empty real points set, which we call *non-standard* and denote as $\mathbb{S}^2$, the quadric whose real point set is a sphere, $\mathbb{S}^2_{1,0}, \mathbb{S}^2_{2,0}, \mathbb{S}^2_{0,2}$, the sphere with blown up one or two real points, or a pair of conjugate imaginary points, respectively, and, at last, $(\mathbb{P}^1)^2_{0,2}$, the standard real quadric blown up at two imaginary conjugate points.

In the present paper we derive the tropical formula for the Welschinger invariants of $\mathbb{S}^2$, $\mathbb{S}^2_{1,0}$, $\mathbb{S}^2_{2,0}$, and $\mathbb{S}^2_{0,2}$[2], that is we express the Welschinger invariants as the sums of weights of certain discrete combinatorial objects, running over specified finite sets, and which are related to tropical curves corresponding to the real algebraic curves in count. The surface $(\mathbb{P}^1)^2_{0,2}$ requires a completely different treatment, and it will be addressed in a forthcoming paper.

The formulation of our result is split into Theorem 1.1, section 1.2.2 (all the surfaces and totally real configurations points), Theorem 1.2, section 1.3.1 (surfaces $\mathbb{S}^2$, $\mathbb{S}^2_{1,0}$, $\mathbb{S}^2_{2,0}$ and configurations with imaginary points), and Theorem 1.3, section 1.3.2 (surface $\mathbb{S}^2_{0,2}$ and configurations with imaginary points). The reason to have a separate statement for the case of totally real configurations is a simpler formulation and its special importance in applications. For example, we prove the positivity of the Welschinger invariants, related to the totally real configurations, which immediately implies the existence of real rational curves belonging to given linear systems and passing through generic configurations of suitable number of real points (see Corollary 1, section 1.2.2).

As compared with the standard real toric Del Pezzo surfaces [14, 17, 18], the case considered in the present paper looks much more degenerate, for instance, the plane tropical curves (amoebas) in count are highly reducible. On the other hand, similarly to the standard case, the final answer is basically expressed as the weighted number of irreducible rational parameterizations of the above plane tropical curves, though the parameterizations generically are not trivalent.

Finally, we notice that, in a joint work with I. Itenberg and V. Kharlamov [11], using Theorem 1.1 we prove the asymptotic relation

$$\lim_{n\to\infty}\frac{\log W_0(\Sigma,\mathcal{L}^{\otimes n})}{n\log n}=\lim_{n\to\infty}\frac{\log N_0(\Sigma,\mathcal{L}^{\otimes n})}{n\log n}=-c_1(\mathcal{L})K_\Sigma\ ,$$

for any real ample line bundle $\mathcal{L}$ on a surface $\Sigma=\mathbb{S}^2$, $\mathbb{S}^2_{1,0}$, $\mathbb{S}^2_{2,0}$, or $\mathbb{S}^2_{0,2}$, which before was established for the standard real toric Del Pezzo surfaces [9].

Welschinger invariants. For the reader's convenience, we recall the definition of

[2]On the real toric Del Pezzo surfaces with empty real point set there are no real rational curves, and thus, Welschinger invariants vanish.

Welschinger invariants. Let Σ be a real toric Del Pezzo surface with a non-empty real part, $\mathcal{L}$ a very ample real line bundle on Σ, and let non-negative integers r', r'' satisfy

$$r' + 2r'' = -c_1(\mathcal{L})K_\Sigma - 1 \ . \tag{0.1}$$

Denote by $\Omega_{r''}(\Sigma, \mathcal{L})$ the set of configurations of $-c_1(\mathcal{L})K_\Sigma - 1$ distinct points of Σ such that r' of them are real and the rest consists of r'' pairs of imaginary conjugate points. The Welschinger number $W_{r''}(\Sigma, \mathcal{L})$ is the sum of weights of all the real rational curves in the linear system $|\mathcal{L}|$, passing through a generic configuration $\overline{\boldsymbol{p}} \in \Omega_{r''}(\Sigma, \mathcal{L})$, where the weight of a real rational curve C is 1 if it has an even number of real solitary nodes, and is -1 otherwise. Since the complex structure of Σ determines a symplectic structure, which is generic in its deformation class, by Welschinger's theorem [19, 20], $W_{r''}(\Sigma, \mathcal{L})$ does not depend on the choice of a generic element $\overline{\boldsymbol{p}} \in \Omega_{r''}(\Sigma, \mathcal{L})$ (a simple proof of this fact can be found in [8]). The definition immediately implies the inequality

$$|W_{r''}(\Sigma, \mathcal{L})| \le R_{\Sigma,\mathcal{L}}(\overline{\boldsymbol{p}}) \le N_{\Sigma,\mathcal{L}} \ , \tag{0.2}$$

where $R_{\Sigma,\mathcal{L}}(\overline{\boldsymbol{p}})$ is the number of real rational curves in $|\mathcal{L}|$ passing through a generic configuration $\overline{\boldsymbol{p}} \in \Omega_{r''}(\Sigma, \mathcal{L})$, and $N_{\Sigma,\mathcal{L}}$ is the number of complex rational curves in $|\mathcal{L}|$, passing through generic $-c_1(\mathcal{L})K_\Sigma - 1$ points in Σ.

A tropical calculation of the Welschinger invariant. Our approach to calculating the Welschinger invariants is quite similar to that in [7, 18], and it heavily relies on the enumerative tropical algebraic geometry developed in [13, 14, 17]. More precisely, we replace the complex field $\mathbb{C}$ by the field $\mathbb{K} = \bigcup_{m\ge1} \mathbb{C}\{\{t^{1/m}\}\}$ of the complex, locally convergent Puiseux series endowed with the standard complex conjugation and with a non-Archimedean valuation

$$\mathrm{Val} : \mathbb{K}^* \to \mathbb{R}, \quad \mathrm{Val}\left(\sum_k a_k t^k\right) = -\min\{k \ : \ a_k \ne 0\} \ .$$

A rational curve over $\mathbb{K}_\mathbb{R}$, belonging to a linear system $|\mathcal{L}|_\mathbb{K}$ and passing through a generic configuration $\overline{\boldsymbol{p}} \in \Omega_{r''}(\Sigma(\mathbb{K}), \mathcal{L})$, is viewed as an equisingular family of real rational curves in Σ over the punctured disc. We construct an appropriate limit of the family of surfaces and embedded curves at the disc center. The central surface is usually reducible, and the adjacency of its components is encoded by a tropical curve in the real plane, which passes through the configuration $\mathrm{Val}(\overline{\boldsymbol{p}}) \subset \mathbb{R}^2$. The central curve is split into components called *limit curves*. The pair (tropical curve, limit curves) is called the]it tropical limit of the given curve $C \in |\mathcal{L}|_\mathbb{K}$.

We precisely describe the tropical limits of real rational curves passing through generic configurations of real points in $\Sigma(\mathbb{K})$, then compute the *Welschinger weights* of the respective tropical curves, i.e., the contribution to the Welschinger invariant of the real algebraic curves projecting to the given tropical curve. The result, accumulated in Theorem 1.1 (section 1.2), represents the Welschinger invariants $W_0(\Sigma, \mathcal{L})$ as the numbers of some combinatorial objects, forming finite discrete sets.

The proof is based on the techniques of [17, 18], both in the determining tropical limits and in the patchworking construction, which recovers algebraic curves over $\mathbb{K}$ from their tropical limits. We should like to remark that the answer rather differs from that for the standard real Del Pezzo surfaces. Namely, in the standard case, the tropical limits are basically encoded by tropical curves, which are rational and irreducible. In the non-standard case, one obtains a relatively small number of possible tropical curves, which split into unions of some primitive tropical curves. In contrast, the weights of the tropical curves are large and are defined in a non-trivial combinatorial way.

We also notice that the patchworking theorems from [17, 18] cover our needs in the present paper. In contrast, the determination of tropical limits meets extra difficulties, caused by the fact that the *generic* configurations of real points on the surfaces under consideration project by $\mathrm{Val} : (\mathbb{K}^*)^2 \to \mathbb{R}^2$ to *non-generic* configurations in $\mathbb{R}^2$ (cf. a similar problem in [17]).

Applications to enumerative geometry. From Theorem 1.1 we immediately derive the positivity of the Welschinger invariants in the considered situations, which in view of (0.2) results in Corollary 1, section 1.2, which says that , for any real very ample line bundle $\mathcal{L}$ on a non-standard real toric Del Pezzo surface $\Sigma = \mathbb{S}^2$, $\mathbb{S}^2_{1,0}$, $\mathbb{S}^2_{2,0}$, or $\mathbb{S}^2_{0,2}$ and any generic configuration of $-c_1(\mathcal{L})K_\Sigma - 1$ real points on Σ there exists a real rational curve $C \in |\mathcal{L}|$ passing through the given configuration.

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1 Tropical formula for the Welschinger invariants

1.1 Lattice polygons associated with the non-standard real toric Del Pezzo surfaces

The non-standard real toric Del Pezzo surfaces $\mathbb{S}^2$, $\mathbb{S}^2_{1,0}$, $\mathbb{S}^2_{2,0}$, $\mathbb{S}^2_{0,2}$, and $(\mathbb{P}^1)^2_{0,2}$ can be associated with the following polygons Δ, respectively (see Figure 1):

- a square $\mathrm{Conv}\{(0,0),(d,0),(0,d),(d,d)\}$, $d \ge 1$;
- a pentagon $\mathrm{Conv}\{(0,0),(0,d),(d-d_1,d),(d,d-d_1),(d,0)\}$, $1 \le d_1 < d$;
- a hexagon $\mathrm{Conv}\{(d_2,0),(0,d_2),(0,d),(d-d_1,d),(d,d-d_1),(d,0)\}$, $1 \le d_1 \le d_2 < d$;
- a hexagon $\mathrm{Conv}\{(0,0),(0,d-d_1),(d_1,d),(d,d),(d,d_1),(d_1,0)\}$, $1 \le d_1 < d$;
- a hexagon $\mathrm{Conv}\{(0,0),(d_1-d_3,0),(d_1,d_3),(d_1,d_2),(d_3,d_2),(0,d_2-d_3)\}$, $1 \le d_3 < d_2 \le d_1$.

For the first four surfaces, the conjugation acts in the torus $(\mathbb{C}^*)^2$ by $\mathrm{Conj}(x,y) = (\overline{y},\overline{x})$, and acts in the tautological line bundle $\mathcal{L}_\Delta$, generated by monomials x^iy^j, $(i,j) \in \Delta \cap \mathbb{Z}^2$, by $\mathrm{Conj}_*(a_{ij}x^iy^j) = \overline{a}_{ij}x^jy^i$, $(i,j) \in \Delta$, resembling the reflection of Δ with respect to the bisectrix $\mathcal{B}$ of the positive quadrant[3]. For the fifth surface, the action in $(\mathbb{C}^*)^2$ is $\mathrm{Conj}(x,y) = (1/\overline{x}, 1/\overline{y})$, and the action in $\mathcal{L}_\Delta$ is $\mathrm{Conj}_*(a_{ij}x^iy^j) = \overline{a_{i,j}}x^{d_1-i}y^{d_2-j}$, $(i,j) \in \Delta$, resembling the reflection of Δ with respect to its center.

Observe that $-c_1(\mathcal{L}_\Delta)K_\Sigma = |\partial\Delta|$, the lattice length of $\partial\Delta$.

1.2 Welschinger invariants associated with the totally real configurations of points

1.2.1 Admissible lattice paths and graphs

For any lattice polygon $\delta \subset \mathbb{R}^2$, symmetric with respect to $\mathcal{B}$, denote by $(\partial\delta)_\perp$ the union of the sides of δ, orthogonal to $\mathcal{B}$, and denote by $(\partial\delta)_+$ the union of the sides of $\delta\mathcal{B}_+$ which are not orthogonal to $\mathcal{B}$. By $\mathrm{Tor}((\partial\delta)_\perp)$ (resp., $\mathrm{Tor}((\partial\delta)_+)$) we denote the union of the toric divisors $\mathrm{Tor}(\sigma)$, where $\sigma \subset (\partial\delta)_\perp$ (resp., $\sigma \subset (\partial\delta)_+$), in the toric surface $\mathrm{Tor}(\delta)$, associated with the polygon δ.

[3]We shall denote by $\mathcal{B}$ the bisectrix of the positive quadrant both in the plane of exponents of the polynomials in consideration, and in the image-plane of the non-Archimedean valuation, no confusion will arise. Furthermore, we shall denote by $\mathcal{B}_+$ and $\mathcal{B}_-$ the halfplanes supported at $\mathcal{B}$ and lying respectively above or below $\mathcal{B}$.

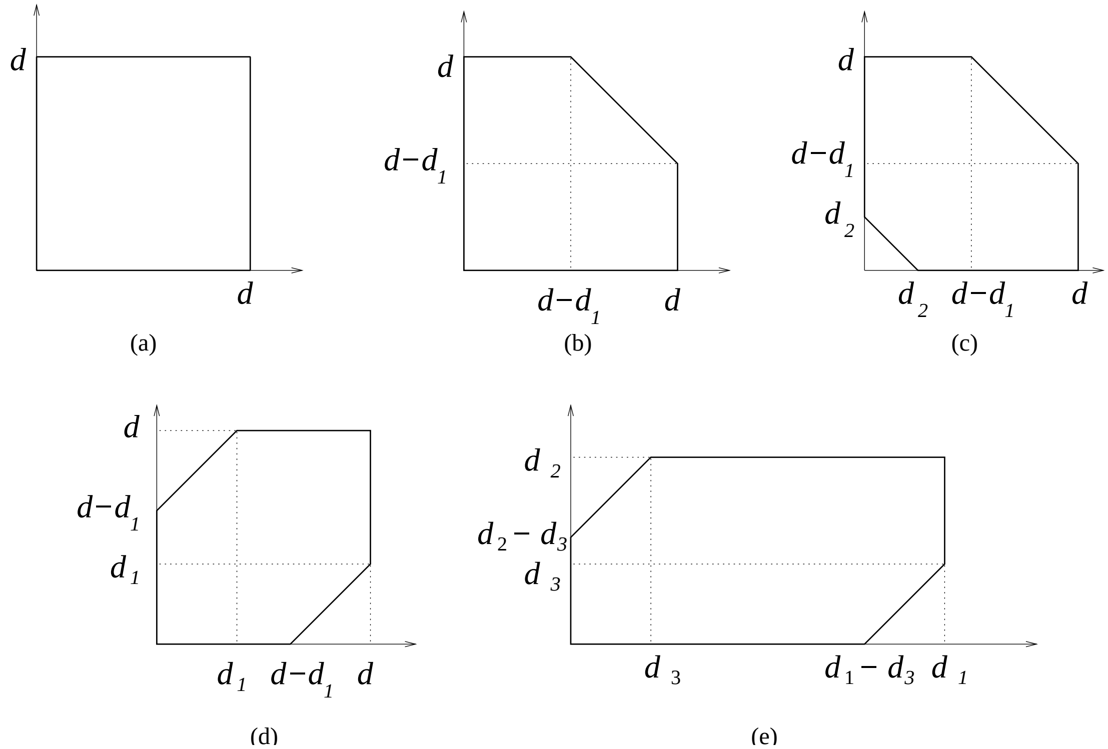


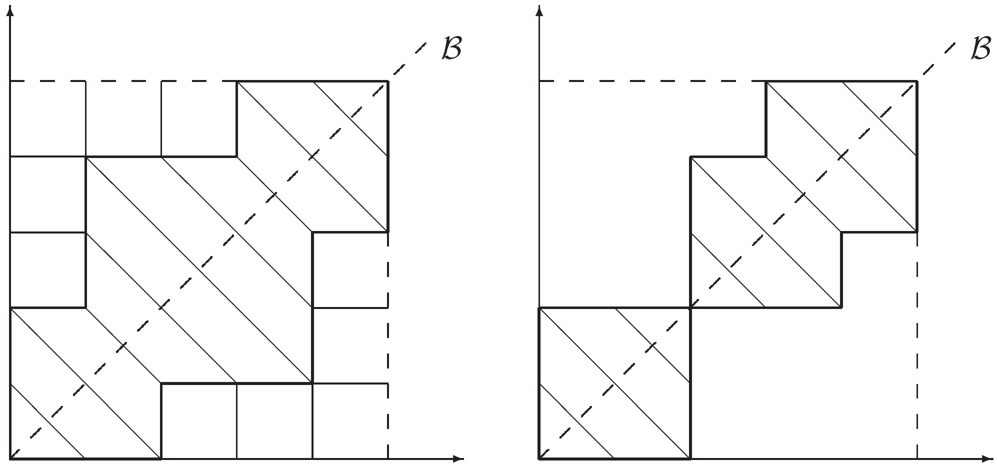
Figure 1: Polygons associated with $\mathbb{S}^2$, $\mathbb{S}^2_{1,0}$, $\mathbb{S}^2_{2,0}$, $\mathbb{S}^2_{0,2}$, and $(\mathbb{P}^1)^2_{0,2}$

Let Δ be one of the four polygons shown in Figure 1(a-d). The integral points divide $(\partial\Delta)_+$ into segments s_i, $1 \le i \le m := |(\partial\Delta)_+|$. An **admissible lattice path** in Δ is a map $\gamma : [0, m] \to \Delta$ such that (see example in Figure 2(a))

- image of γ lies in $\mathcal{B}_+$;
- $\gamma(0)$ and $\gamma(m)$ are the two endpoints of $(\partial\Delta)_+$;
- the composition of the functional $x+y$ with γ is a strongly increasing function;
- $\gamma(i) \in \mathbb{Z}^2$, and $\gamma\big|_{[i,i+1]}$ is linear as $i \in \mathbb{Z}$;
- there is a permutation $\tau \in S_{m-1}$ such that $\gamma([i-1, i])$ is a translate of the segment $s_{\tau(i)}$, $1 \le i \le m$, of $(\partial\Delta)_+$;
- $\gamma([0, m]) \cap \mathcal{B} = (\partial\Delta)_+ \cap \mathcal{B}$.

The image of γ completely determines the map, and we shall denote them by the same symbol γ.

Denote by σ_i, $0 \le i \le m$, the segments, joining the integral points of γ with their symmetric images (see Figure 2(a)).



(a) (b)

Figure 2: Lattice paths and subdivisions of Δ

A **γ-admissible graph** G is defined as follows. First, we describe some subgraph G'. The connected components of G' are lattice segments (or points) $G'_j = [(a_j, j), (b_j, j)] \subset \mathbb{R}^2$, $1 \le j \le n := |\partial\Delta| - m - 1$, with positive integer weights $w(G'_j)$ such that

- $0 \le a_j \le b_j \le m$ for all $j = 1, ..., n$;
- $a_j \le a_{j+1}$, and in addition $b_j \le b_{j+1}$ if $a_j = a_{j+1}$ as $j = 1, ..., n-1$;
- for all $i = 0, ..., m$,
$$\sum_{(i,j)\in G'_j} w(G'_j) = |\sigma_i| \ ; \tag{1.3}$$
- if $a_j = 0$ or $b_j = m$ then $w(G'_j) = 1$.

We then introduce new vertices φ_i, $i = 1, ..., m$, of the graph G and the new arcs, joining any vertex φ_i with the endpoint $(i-1, j)$ of any component G'_j satisfying $b_j = i+1$, and with the endpoint (i, j) of any component G'_j satisfying $a_j = i$. Our final requirement is that the obtained graph G is a tree.

A **marking** of a γ-admissible graph G is a vector $\overline{s} = (s_1, ..., s_n) \in \mathbb{Z}^n$ such that $a_j \le s_j \le b_j$, $j = 1, ..., n$, subject to the following restriction:

$$s_j \le s_{j+1} \quad \text{as far as} \quad a_j = a_{j+1},\ b_j = b_{j+1} \ .$$

At last we define the **Welschinger number** $W(\gamma, G, \overline{s}) = 0$ if at leat one weight $w(G_i')$ is even, and otherwise

$$W(\gamma, G, \overline{s}) = 2^v \prod_{k=0}^{m} n_k! \left(\prod_{\substack{0 \le a \le b \le m \\ c=1,3,5,...}} n_{k,a,b,c}! \right)^{-1} , \tag{1.4}$$

where v is the total valency of those vertices φ_i of G, for which $|\sigma_i| = |\sigma_{i-1}|$, and

$$n_k := \#\{j \,:\, s_j = k,\, 1 \le j \le n\}, \quad n_{k,a,b,c} = \#\{j \,:\, s_j = k,\, a_j = a,\, b_j = b,\, w(G_j') = c\},$$

$$k = 0, ..., m, \quad 0 \le a \le k \le b \le m, \quad c = 1, 3, 5, ...$$

1.2.2 The formula

Theorem 1.1 *In the notation of sections 1.1 and 1.2.1, if $\Sigma = \mathbb{S}^2$, $\mathbb{S}^2_{1,0}$, $\mathbb{S}^2_{2,0}$, or $\mathbb{S}^2_{0,2}$, then*

$$W_0(\Sigma, \mathcal{L}_\Delta) = \sum W(\gamma, G, \overline{s}) \ , \tag{1.5}$$

where the sum ranges over all admissible lattice paths γ, all γ-admissible graphs G, and all markings $\overline{s}$ of G.

It is an easy exercise to show that there always exist an admissible lattice path and a corresponding admissible graph, and hence

Corollary 1 *In the above notation, for any surface $\Sigma = \mathbb{S}^2$, $\mathbb{S}^2_{1,0}$, $\mathbb{S}^2_{2,0}$, or $\mathbb{S}^2_{0,2}$, and any line bundle $\mathcal{L}_\Delta$, the Welschinger invariant $W_0(\Sigma, \mathcal{L}_\Delta)$ is positive, and through any $-c_1(\mathcal{L}_\Delta)K_\Sigma - 1$ generic real points on Σ there passes at least one real rational curve $D \in |\mathcal{L}_\Delta|$.*

1.2.3 Examples

(A) Linear systems with an elliptic general member. Let $\Sigma = \mathbb{S}^2$, $\mathbb{S}^2_{1,0}$, $\mathbb{S}^2_{2,0}$, or $\mathbb{S}^2_{0,2}$, and let Δ be a respective associated lattice polygon as shown in Figure 1 and such that a general curve in $|\mathcal{L}_\Delta|$ is elliptic. Then Δ is as depicted in Figure 3.

The Welschinger invariant $W_0(\Sigma, \mathcal{L}_\Delta)$ can be computed by counting rational curves in the pencil of real curves in $|\mathcal{L}_\Delta|$ passing through $(-c_1(\mathcal{L}_\Delta)K_\Sigma - 1)$ generic real points. Integrating along the pencil with respect to the Euler characteristic and noticing that the curves in the pencil have one more real base point, and $\chi(\mathbb{R}D) = 1$ or -1 according as $D \in |\mathcal{L}_\Delta|$ is a real rational curve with a solitary or a non-solitary node, we obtain (cf. with the case of plane cubics [9], section 3.1)

$$W_0(\Sigma, \mathcal{L}_\Delta) = -c_1(\mathcal{L}_\Delta)K_\Sigma - \chi(\Sigma(\mathbb{R})) \ ,$$

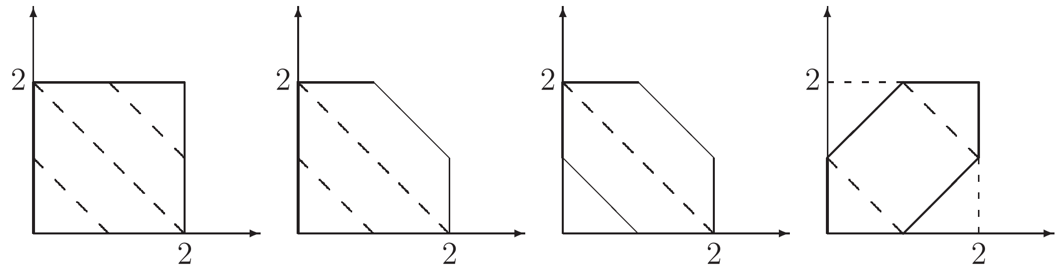


Figure 3: Linear systems with an elliptic general curve

which equals 6, 6, 6, or 4 as $\Sigma = \mathbb{S}^2$, $\mathbb{S}^2_{1,0}$, $\mathbb{S}^2_{2,0}$, or $\mathbb{S}^2_{0,2}$, respectively.

In turn in Theorem 1.1 we have a unique admissible path γ (fat line in Figure 3) and a unique subdivision of Δ (dashes in Figure 3). The subgraphs G' of the γ-admissible graphs, their markings and Welschinger numbers are shown in Figure 4 (the weight of any component of G' is here 1). The result, of course, coincides with the aforementioned one.

(B) Linear systems of digonal curves. We illustrate Theorem 1.1 by two more examples, where one can easily write down a closed formula for the Welschinger invariant (a similar computation has been performed for digonal curves on $(\mathbb{P}^1)^2$ [9], section 3.1). Namely, we consider the surfaces $\Sigma = \mathbb{S}^2_{2,0}$ and $\mathbb{S}^2_{0,2}$ and the linear systems associated with the polygons Δ shown in Figure 5(a,b), respectively.

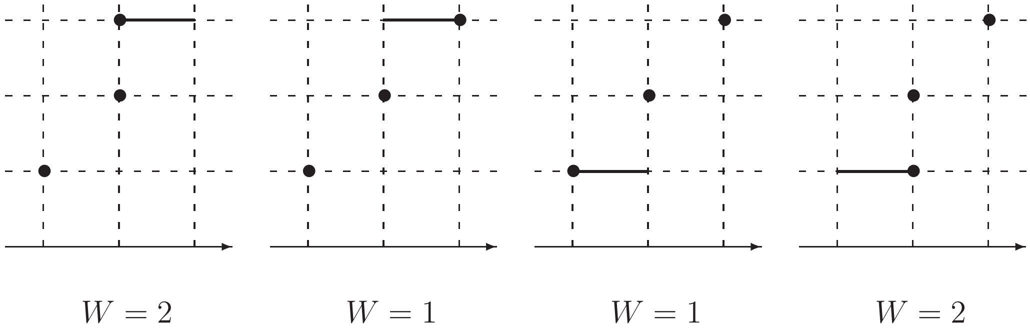
In the case $\Sigma = \mathbb{S}^2_{0,2}$ (see Figure 5(b)) we have a unique admissible path γ going just along $(\partial\Delta)_+$, a unique γ-admissible graph G, and a unique marking (see Figure 5(e)). Hence we obtain $W_0(\mathbb{S}^2_{0,2}, \mathcal{L}_\Delta) = 4^{d-1}$, d being the length of projection of Δ on a coordinate axis.

In the case $\Sigma = \mathbb{S}^2_{2,0}$, $d > 2$, there are two admissible lattice paths γ_1, γ_2 (shown by fat lines in Figure 5(c,d)). The subgraph G' of an admissible graph G, and a marking $\overline{s}$ should look as shown in Figure 5(f), where we denote by k (resp., l) the number of components $[(1,j),(2,j)]$ (resp. $[(2,j),(3,j)]$), and k_1 (resp. l_1) is the number of the marked points $(2,j)$ on components $[(1,j),(2,j)]$ (resp. $[(2,j),(3,j)]$), and where (k,l,k_1,l_1) run over the sets $J(\gamma_1)$ and $J(\gamma_2)$ defined by

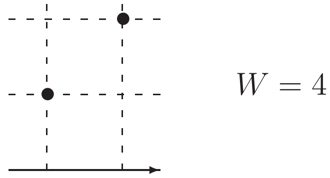
$$0 \le k_1 \le k \le d-1, \quad 0 \le l_1 \le l \le d-1, \quad d-k-l = 2a+1, \ a \ge \begin{cases} 1, & \text{if } \gamma = \gamma_1, \\ 2, & \text{if } \gamma = \gamma_2 \end{cases}$$

Here the weights of all the components of G' are equal to 1, except for the one-point component on the middle vertical line, whose weight is $2a+1$ or $2a-1$ according as $\gamma = \gamma_1$ or γ_2. Thus, we obtain

$$W_0(\mathbb{S}^2_{2,0}, \mathcal{L}_\Delta) = \sum_{i=1}^{2} \sum_{(k,l,k_1,l_1)\in J(\gamma_i)} \frac{(d-1-k_1)!(d-1-l_1)!(k_1+l_1+1)!}{(d-1-k)!(d-1-l)!(k-k_1)!(l-l_1)!k_1!l_1!} \ .$$



(a) $\Sigma = \mathbb{S}^2,\ \mathbb{S}^2_{1.0},\ \text{or}\ \mathbb{S}^2_{2,0}$



(b) $\Sigma = \mathbb{S}^2_{0,2}$

Figure 4: Admissible graphs and markings, I

1.3 Welschinger invariants associated with configurations containing imaginary points

Let Σ be one of the surfaces $\mathbb{S}^2$, $\mathbb{S}^2_{1,0}$, $\mathbb{S}^2_{2,0}$, or $\mathbb{S}^2_{0,2}$, and let Δ be a respective lattice polygon shown in Figure 1(a-d). Fix positive integers r', r'' satisfying (0.1) in the form

$$r' + 2r'' = |\partial\Delta| - 1 \ . \tag{1.6}$$

1.3.1 The case of $\Sigma = \mathbb{S}^2$, $\mathbb{S}^2_{1,0}$, or $\mathbb{S}^2_{2,0}$

First, we introduce a splitting $\mathcal{R}$ of r' and r'' into nonnegative integer summands

$$r' = r'_1 + r'_2, \quad r'' = r''_1 + r''_{2,1} + r''_{2,2} \ , \tag{1.7}$$

so that (cf. (1.6), (1.7))

$$r'_1 + 2r''_1 + r''_{2,1} = |\partial\Delta| - |\partial\Delta|_+ - 1, \quad r'_2 + r''_{2,1} + 2r''_{2,2} = |\Delta|_+ \ . \tag{1.8}$$

Notice here that, for $r'' = 0$, the splitting $\mathcal{R}$ turns into

$$r' = |\partial\Delta| - 1, \quad r'_1 = |\partial\Delta| - |\partial\Delta|_+ - 1, \quad r'_2 = |\Delta|_+ \ . \tag{1.9}$$

Take a lattice path γ in Δ, admissible in the sense of section 1.2.1. We then pick integral points v_i, $i = 0, 1, ..., \widetilde{m} := r'_2 + r''_{2,1} + r''_{2,2}$, on γ such that

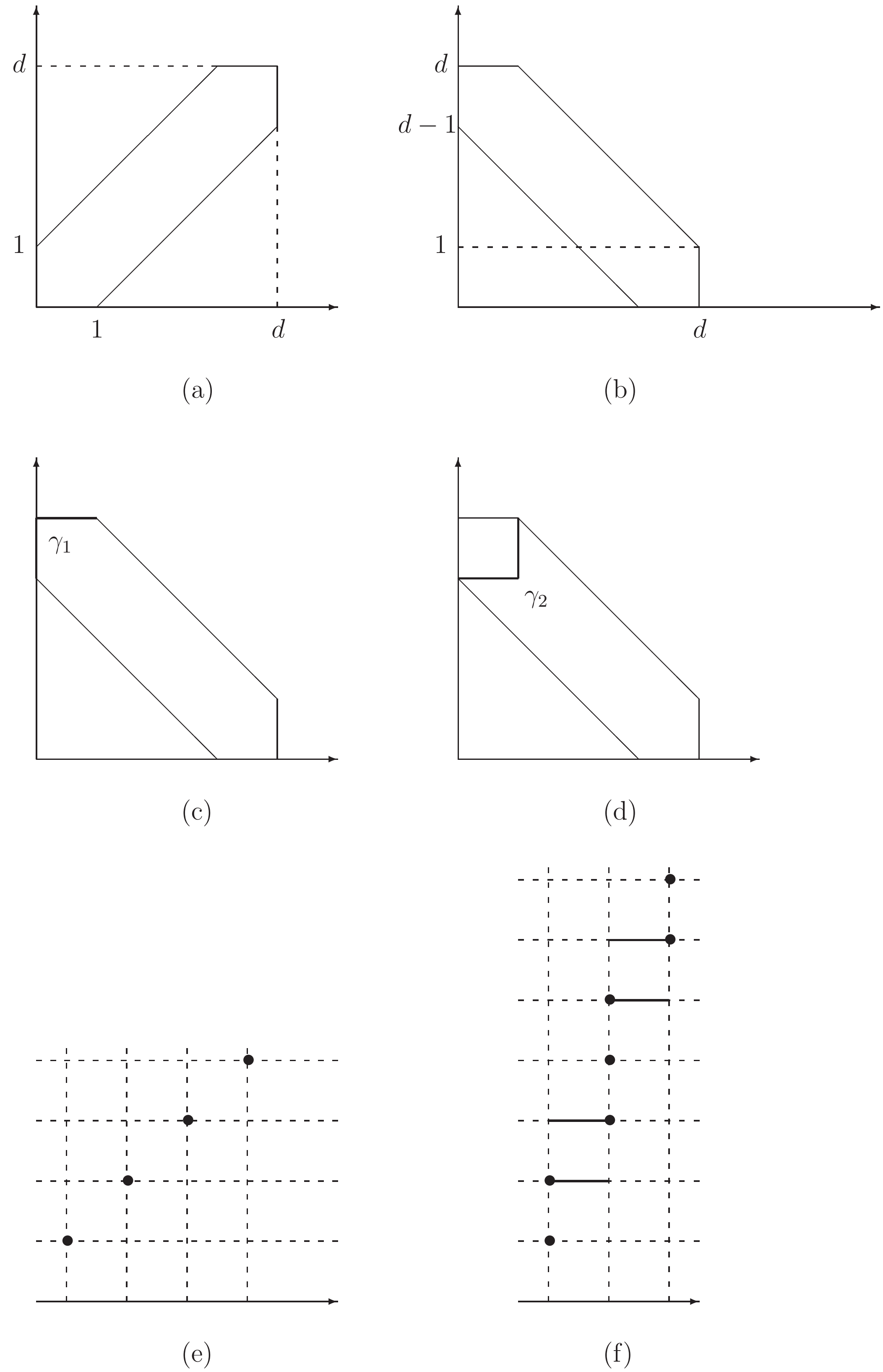


Figure 5: Admissible paths, graphs, and markings, II

(γ**1**) they are ordered by the growing sum of coordinates;

(γ**2**) v_0 and $v_{\widetilde{m}}$ are the endpoints;

(γ**3**) the lattice length of the part γ_i of γ between the points v_{i-1}, v_i, $1 \le i \le \widetilde{m}$, is 1 or 2;

(γ**4**) $|\gamma_i| = 1$ for all $i > r''_{2,1} + r''_{2,2}$;

(γ**5**) each break point of γ, where the path turns in positive direction (i.e, left), is one of v_i, $1 \le i \le \widetilde{m}$.

Denote by σ_i the segment, joining v_i with its symmetric (with respect to $\mathcal{B}$) image, $i = 0, ..., \widetilde{m}$. Observe that γ defines a lattice subdivision S of Δ, consisting of the polygons in the part of Δ between γ and its symmetric image $\hat{\gamma}$, cut out by the segments σ_i, $i = 1, ..., \widetilde{m} - 1$, and the rectangles with vertical and horizontal sides in the remaining part of Δ (see Figure 2(a)).

Then we construct a $(\gamma, \mathcal{R})$-admissible graph G, starting with an auxiliary subgraph G', whose connected components are segments or points $G'_j = [(a_j, j), (b_j, j)]$ lying on the lines $y = j$ with $j = 1, ..., n := |\partial\Delta| - |\partial\Delta|_+ - 1 = r'_1 + 2r''_1 + r''_{2,1}$, respectively, equipped with positive integer weights $w(G'_j)$, and satisfying the following conditions:

(**G1**) for all $j = 1, ..., n$,
$$0 \le a_j \le b_j \le \widetilde{m} \ ;$$

(**G2**) for all $j = 1, ..., r''_1$,
$$a_{2j-1} = a_{2j}, \quad b_{2j-1} = b_{2j}, \quad w(G'_{2j-1}) = w(G'_{2j}) \ ;$$

(**G3**) $w(G'_j) = 1$ as far as $a_j = 0$ or $b_j = \widetilde{m}$;

(**G4**) $a_j \le a_{j+1}$ and, if $a_j = a_{j+1}$ then $b_j \le b_{j+1}$, as far as either $1 \le j < 2r''_1$, or $2r''_{2,1} < j < 2r''_1 + r''_{2,1}$, or $2r''_1 + r''_{2,1} < j < n$;

(**G5**) if $|\gamma_i| = 2$ for some $1 \le i \le \widetilde{m}$, then
$$a_j \ne i + 1 \text{ and } b_j \ne i \quad \text{for all } 2r''_1 < j \le n \ ;$$

(**G6**) $a_j \le r''_{2,1} + r''_{2,2}$ for $2r''_1 < j \le 2r''_1 + r''_{2,1}$;

(**G7**) for any $i = 0, ..., \widetilde{m}$, relation (1.3) holds true.

Next we introduce some more vertices of G, taking one vertex φ_i for all $i = 1, ..., \widetilde{m}$ such that $|\gamma_i| = 1$, and taking two vertices $\varphi_{i,1}, \varphi_{i,2}$ for all $i = 1, ..., \widetilde{m}$ such that $|\gamma_i| = 2$, and, finally, we define additional arcs of G as follows:

(**G8**) any vertex φ_i, $1 \le i \le \widetilde{m}$, is joined by arcs with each endpoint (a_j, j), $a_j = i$, of a component of G', and with each endpoint (b_j, j), $b_j = i-1$, of a component of G';

(**G9**) any vertex $\varphi_{i,1}$ (resp., $\varphi_{i,2}$) is joined by arcs with each endpoint $(a_{2j-1}, 2j-1)$ (resp., $(a_{2j}, 2j)$), $a_{2j-1} = i$, $1 \le j \le r''_1$, of a component of G', and with each endpoint $(b_{2j-1}, 2j-1)$ (resp., $(b_{2j}, 2j)$), $b_{2j} = i-1$, $1 \le j \le r''_1$, of a component of G';

(**G10**) the minimal subgraph G'' of G, containing the components G'_j, $2r''_1 < j \le 2r''_1 + r''_{2,1}$, and all the vertices φ_i, $1 \le i \le r''_{2,1} + r''_{2,2}$, is a forest, each component of G'' contains at least one univalent vertex of type φ_i, furthermore, any component of G'' can be oriented so that from each vertex of type φ_i emanates precisely one oriented arc, and no arc emanates from a vertex of type $(0, j)$ or $(\widetilde{m}, j)$;

(**G11**) G is a tree.

A marking of a $(\gamma, \mathcal{R})$-admissible graph G is a pair of integer vectors $\overline{s}' = (s'_1, ..., s'_{r'_1})$ and $\overline{s}'' = (s''_1, ..., s''_{r''_1})$ such that

(**s1**) $a_{j+2r''_1+r''_{2,1}} \le s'_j \le b_{j+2r''_1+r''_{2,1}}$ as $j = 1, ..., r'_1$, and $a_{2j} \le s''_j \le b_{2j}$ as $j = 1, ..., r''_1$;

(**s2**) $s'_j \le s'_{j+1}$ as far as $a_{j+2r''_1+r''_{2,1}} = a_{j+2r''_1+r''_{2,1}+1}$, $b_{j+2r''_1+r''_{2,1}} = b_{j+2r''_1+r''_{2,1}+1}$, where $1 \le j < r'_1$;

(**s3**) $s''_j \le s''_{j+1}$ as far as $a_{2j} = a_{2j+1}$, $b_{2j} = b_{2j+1}$, where $1 \le j < r''_1$;

(**s4**) $s'_j \ge s''_k$ for all $j = 1, ..., r'_1$ and $k = 1, ..., r''_1$.

Observe that condition (**s4**) imposes an extra restriction to the components of G' and splitting $\mathcal{R}$.

Then we define the Welschinger number as $W(\mathcal{R}, \gamma, G, \overline{s}', \overline{s}'') = 0$ if at least one weight $w(G'_i)$, $2r''_1 < i \le n$, is even, and otherwise as

$$W(\mathcal{R}, \gamma, G, \overline{s}', \overline{s}'') = (-1)^a \cdot 2^{b+c} \cdot \prod_{j=1}^{r''_1} (w(G'_{2j}))^2 \cdot \prod_{j=2r''_1+1}^{2r''_1+r''_{2,1}} w(G'_j) \cdot \prod_{i=0}^{\widetilde{m}} \left(n'_i! n''_i! \alpha_i^{-1} \beta_i^{-1}\right) \ , \tag{1.10}$$

$$\alpha_i = \prod_{\substack{0\le d\le e\le\widetilde{m}\\ f=1,3,5,...}} n'_{i,d,e,f}!, \quad \beta_i = \prod_{\substack{0\le d\le e\le\widetilde{m}\\ f=1,2,3,...}} n''_{i,d,e,f}! \ ,$$

where

- $a = (a_1 - a'_1) + ... + (a_{\widetilde{m}} - a'_{\widetilde{m}})$ with the following summands: if $|\gamma_k| = 1$, then $a_k = a'_k = 0$, if $|\gamma_k| = 2$, then a_k is the length of the segment $[p, \hat{p}]$, p being the intermediate integral point of γ, and a'_k is the total weight of the components of G', crossing the vertical line $x = k - 1/2$;
- b is the total valency of all the vertices $\varphi_{i,1}$ of G;
- c is the number of hexagons in the subdivision S;
- $n'_i = \#\{j \ : \ s'_j = i, \ 1 \le j \le r'_1\}$, $n''_i = \#\{j \ : \ s''_j = i, \ 1 \le j \le r''_1\}$;
- $n'_{i,d,e,f} = \#\{j \ : \ s'_j = i, \ a_j = d, \ b_j = e, \ w(G'_j) = f\}$;
- $n''_{i,d,e,f} = \#\{j \ : \ s''_j = i, \ a_{2j} = d, \ b_{2j} = e, \ w(G'_{2j}) = f\}$.

Theorem 1.2 *In the notation of section 1.3.1, if $\Sigma = \mathbb{S}^2$, $\mathbb{S}^2_{1,0}$, or $\mathbb{S}^2_{2,0}$, and the positive integers r', r'' satisfy (1.6), then*

$$W_{r''}(\Sigma, \mathcal{L}_\Delta) = \sum W(\mathcal{R}, \gamma, G, \overline{s}', \overline{s}'') \ , \tag{1.11}$$

where the sum ranges over splittings $\mathcal{R}$ of r', r'', satisfying (1.8), all admissible lattice paths γ, all γ-admissible graphs G, and all markings $\overline{s}', \overline{s}''$ of G, subject to conditions, specified in section 1.3.1.

As example we consider the linear system of curves of bi-degree $(2, 2)$ on $\mathbb{S}^2$. Here $r' + 2r'' = 7$, and the integration with respect to Euler characteristic (cf. section 1.2.3, case (A)) gives $W_{r''}(\mathbb{S}^2, (2, 2)) = 6 - 2r''$, $r'' = 1, 2, 3$. In Figure 6 we demonstrate how to obtain these answers from formula (1.11), listing admissible paths γ, subdivisions S of Δ, and graphs G' (here the marked points are denoted by bullets and the non-marked one-point components of G' are denoted by circles).

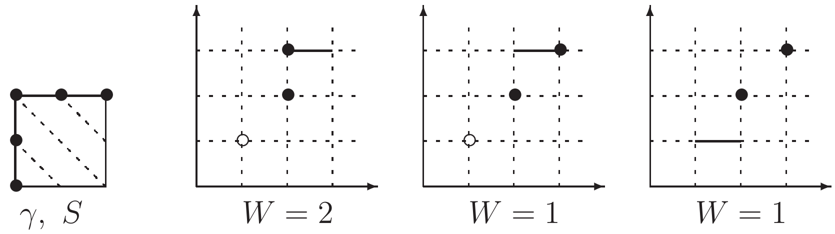
1.3.2 The case of $\Sigma = \mathbb{S}^2_{0,2}$

Let $\mathcal{R}$ be a splitting of r' and r'' as in (1.7) so that

$$r'_2 + r''_{2,1} + 2r''_{2,2} = m := |(\partial\Delta)_+| + n_0 \tag{1.12}$$

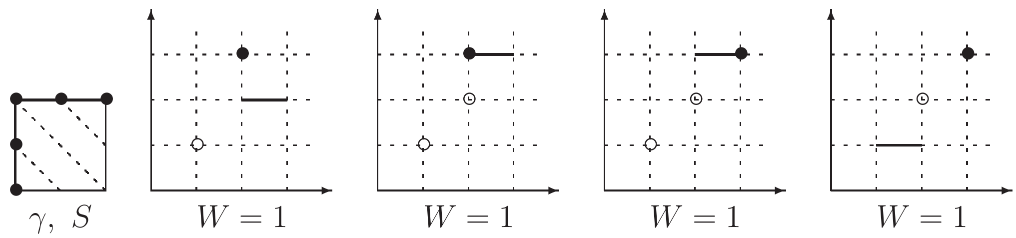
with some $0 \le n_0 \le d_1$. We remark that, for $r'' = 0$, necessarily $n_0 = 0$ (see Remark 2.3 in section 2.3 below), and the splitting $\mathcal{R}$ turns into (1.9).

An $\mathcal{R}$-admissible lattice path in Δ is a map $\gamma : [0, m] \to \Delta$ such that

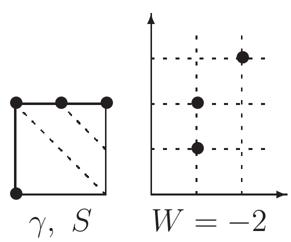


$\mathcal{R}=\{r'=2+3,\ r''=0+1+0\},\quad \widetilde{m}=4,\ n=3$

(a) Case $\quad r'=5,\ r''=1,\ W=4$

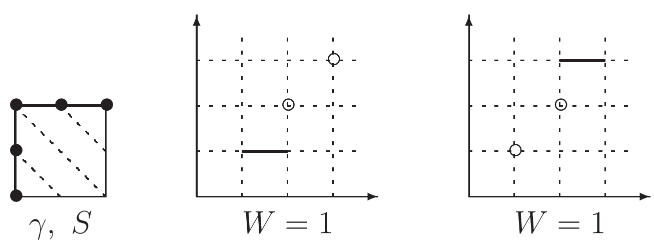


$\mathcal{R}=\{r'=1+2,\ r''=0+2+0\},\quad \widetilde{m}=4,\ n=3$

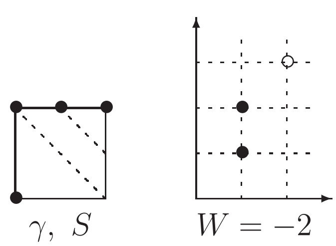


$\mathcal{R}=\{r'=1+2,\ r''=1+0+1\}$
$\widetilde{m}=3,\ n=3$

(b) Case $\quad r'=3,\ r''=2,\ W=2$



$\mathcal{R}=\{r'=0+1,\ r''=0+3+0\}$
$\widetilde{m}=4,\ n=3$



$\mathcal{R}=\{r'=0+1,\ r''=1+1+1\}$
$m=3,\ n=3$

(c) Case $\quad r'=1,\ r''=3,\ W=0$

Figure 6: Admissible paths, graphs, and markings, III

(γ**1'**) image of γ lies in $\mathcal{B}_+$;

(γ**2'**) $\gamma(0)$ and $\gamma(m)$ are the two endpoints of $(\partial\Delta)_+$;

(γ**3'**) the composition of the functional $x+y$ with γ is a strongly increasing function;

(γ**4'**) for any $i = 1, ..., m$, the point $\gamma(i)$ is integral, the function $\gamma\big|_{[i-1,i]}$ is linear, and $\gamma[i-1,i]$ is a unit length segment, whose projection on $\mathcal{B}$ has lattice length $1/2$ if it is not parallel to $\mathcal{B}$;

(γ**5'**) $\gamma([0,m]) \cap \mathcal{B}$ is finite.

We then pick integral points v_i, $i = 0, 1, ..., \widetilde{m} := r'_2 + r''_{2,1} + r''_{2,2}$, on γ, satisfying conditions (γ**1**)-(γ**5**) from section 1.3.1 and the condition

(γ**6'**) each break point of γ, which belongs to a segment parallel to $\mathcal{B}$, is among v_i, $1 \le i \le \widetilde{m}$.

Denote by σ_i the segment, joining v_i with its symmetric (with respect to $\mathcal{B}$) image, $i = 1, ..., \widetilde{m} - 1$.

A lattice path γ with points $v_1, ..., v_{\widetilde{m}}$ gives rise to a set (may be, empty) of γ-admissible lattice subdivisions S of Δ, symmetric with respect to $\mathcal{B}$, and resulting from the following construction:

(**S1**) the part of Δ between γ and its symmetric with respect to $\mathcal{B}$ image $\hat{\gamma}$ is subdivided by the segments σ_i, $i = 1, ..., \widetilde{m} - 1$ (see, for instance, Figure 2);

(**S2**) denote by v_i, $\widetilde{m} < i \le \widetilde{m}_1$, all the break points of γ, which do not appear among v_i, $1 \le i \le \widetilde{m}$;

(**S3**) take the first break point v of γ, where γ turns in the positive direction (i.e., left), and denote by $\sigma^{(1)}, \sigma^{(2)}$ the minimal segments of γ, emanating from v and ending up at points v_i, $1 \le i \le \widetilde{m}_1$; then take a new polygon Δ' of S to be either

- the parallelogram, spanned by $\sigma^{(1)}, \sigma^{(2)}$ (see Figure 8(a)), or

- if $|\sigma^{(1)}| = |\sigma^{(2)}| = 1$, the triangle, spanned by $\sigma^{(1)}, \sigma^{(2)}$, with the unit length third side parallel to $\mathcal{B}$ (see Figure 8(b)), or

- if $|\sigma^{(1)}| = 2$, $|\sigma^{(2)}| = 1$ (resp., $|\sigma^{(1)}| = 1$, $|\sigma^{(2)}| = 2$), the trapeze with the unit length sides $\sigma^{(3)}, \widetilde{\sigma}$, correspondingly parallel to $\sigma^{(1)}$, $\mathcal{B}$ (resp., $\sigma^{(2)}$, $\mathcal{B}$) (see Figure 8(d,e)), or

- if $|\sigma^{(1)}| = |\sigma^{(2)}| = 2$, the pentagon with the unit length sides $\sigma^{(3)}, \sigma^{(4)}, \widetilde{\sigma}$ parallel to $\sigma^{(1)}, \sigma^{(2)}, \mathcal{B}$, respectively (see Figure 8(f));

(S4) if $\Delta' \not\subset \Delta$ we stop the procedure, if $\Delta' \subset \Delta$, we introduce the new lattice path γ_1, replacing $\sigma^{(1)}, \sigma^{(2)}$ in γ by $(\partial\Delta')_+$, and denote the new break points of γ_1 by v_i, $\widetilde{m}_1 < i \le \widetilde{m}_2$;

(S5) having a lattice path γ_l and the points v_i, $1 \le i \le \widetilde{m}_{l+1}$, we perform the above steps (**S3**), (**S4**), and proceed inductively until the construction stops;

(S6) in case $\gamma_l = (\partial\Delta)_+$, we reflect the obtained polygons with respect to $\mathcal{B}$ and obtain a γ-admissible subdivision S of Δ.

Observe that such a subdivision S of Δ is dual to a plane tropical curve A (for the definition see, for example, [14], section 3.4, or section 2.1 in the present paper below). The tropical curve A has exactly $\widetilde{m}$ vertices on $\mathcal{B}$, which we denote by $w_1, ..., w_{\widetilde{m}}$ in the growing coordinate order. These vertices are dual to the polygons of S, symmetric with respect to $\mathcal{B}$. There are n_0 polygons $\widetilde{\Delta}_i$, $1 \le i \le n_0$, of S in $\mathcal{B}_+$, different from parallelograms, and the dual vertices of A in $\mathcal{B}_+$ we denote by w_i^+, $1 \le i \le n_0$. Each of them is joined in A with two of the points $w_1, ..., w_{\widetilde{m}}$ by segments.

Then we construct a $(\gamma, \mathcal{R}, S)$-admissible graph G, starting with an auxiliary subgraph G', whose connected components are segments or points $G'_j = [(a_j, j), (b_j, j)]$ lying on the lines $y = j$ with $j = 1, ..., n := r'_1 + 2r''_1 + r''_{2,1}$, respectively, equipped with positive integer weights $w(G'_j)$, satisfying conditions (**G1**)-(**G2**), (**G4**)-(**G6**) and the conditions

(G7') for any $i = 1, ..., \widetilde{m} - 1$, relation (1.3) holds true.

Next we introduce some more vertices of G, taking one vertex φ_i for all $i = 1, ..., \widetilde{m}$ such that $|\gamma_i| = 1$, and taking two vertices $\varphi_{i,1}, \varphi_{i,2}$ for all $i = 1, ..., \widetilde{m}$ such that $|\gamma_i| = 2$, and define new arcs of G following rules (**G8**), (**G9**) and keeping condition (**G10**) of section 1.3.1.

Another set of vertices of G is produced when assigning to each vertex w_i^+ a pair of vertices $\varphi_{i,1}^+, \varphi_{i,2}^+$ of G. Then we introduce additional arcs:

(G12) if $|\gamma_i| = 1$ and the vertex w_i of A is joined by a segment with a vertex w_j^+, then we connect the vertex φ_i of G both with $\varphi_{j,1}^+$ and $\varphi_{j,2}^+$ by arcs;

(G13) if $|\gamma_i| = 2$ and the vertex w_i of A is joined by a segment with only one vertex w_j^+, $1 \le j \le n_0$, then either we connect by arcs the vertex $\varphi_{i,1}$ of G with $\varphi_{j,1}^+$, and the vertex $\varphi_{i,2}$ with $\varphi_{j,2}^+$, or we connect $\varphi_{i,1}$ with $\varphi_{j,2}^+$, and $\varphi_{i,2}$ with $\varphi_{j,1}^+$;

(**G14**) if $|\gamma_i| = 2$ and the vertex w_i of A is joined by segments with two vertices w^+j, w^+_k, then either we connect the vertex $\varphi_{i,1}$ both with $\varphi^+_{j,1}$ and $\varphi^+_{k,2}$, respectively, the vertex $\varphi_{i,2}$ both with $\varphi^+_{j,2}$ and $\varphi^+_{k,1}$, or we connect the vertex $\varphi_{i,1}$ both with $\varphi^+_{j,2}$ and $\varphi^+_{k,1}$, respectively, the vertex $\varphi_{i,2}$ both with $\varphi^+_{j,1}$ and $\varphi^+_{k,2}$.

Our final requirement to G is that is must be a tree (condition (**G11**) of section 1.3.1).

A marking of a $(\gamma, \mathcal{R}, S)$-admissible graph G is a pair of integer vectors $\overline{s}' = (s'_1, ..., s'_{r'_1})$ and $\overline{s}'' = (s''_1, ..., s''_{r''_1})$, satisfying conditions (**s1**)-(**s4**) of section 1.3.1.

We define the Welschinger number as $W(\mathcal{R}, \gamma, G, \overline{s}', \overline{s}'') = 0$ if at least one weight $w(G'_i)$, $2r''_1 < i \leq n$, is even, and otherwise as

$$W(\mathcal{R}, \gamma, S, G, \overline{s}', \overline{s}'') = (-1)^a \cdot 2^{b+c} \cdot \prod_{j=1}^{r''_1} (w(G'_{2j}))^2 \cdot \prod_{j=2r''_1+1}^{2r''_1+r''_{2,1}} w(G'_j)$$

$$\times \prod_{i=0}^{\widetilde{m}} \left(n'_i! n''_i! \alpha_i^{-1} \beta_i^{-1}\right) \cdot \prod_{k=1}^{n_0} \mathrm{Ar}(\widetilde{\Delta}_k) \ , \tag{1.13}$$

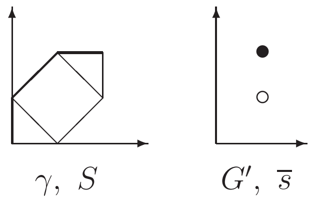
$$\alpha_i = \prod_{\substack{0 \leq d \leq e \leq \widetilde{m} \\ f=1,3,5,...}} n'_{i,d,e,f}!, \quad \beta_i = \prod_{\substack{0 \leq d \leq e \leq \widetilde{m} \\ f=1,2,3,...}} n''_{i,d,e,f}! \ ,$$

where

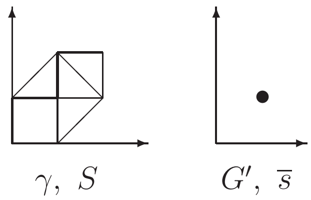
- a, b are as in formula (1.10);
- c is the number of the polygons Δ' in the subdivision S, which are symmetric with respect to $\mathcal{B}$ and whose boundary part $(\partial\Delta')_+$ is not a segment;
- $n'_i = \#\{j \ : \ s'_j = i, \ 1 \leq j \leq r'_1\}$, $n''_i = \#\{j \ : \ s''_j = i, \ 1 \leq j \leq r''_1\}$;
- $n'_{i,d,e,f} = \#\{j \ : \ s'_j = i, \ a_j = d, \ b_j = e, \ w(G'_j) = f\}$;
- $n''_{i,d,e,f} = \#\{j \ : \ s''_j = i, \ a_{2j} = d, \ b_{2j} = e, \ w(G'_{2j}) = f\}$;
- $\mathrm{Ar}(\widetilde{\Delta}_k)$ is the following: $\widetilde{\Delta}_k$ uniquely splits into the Minkowski sum of a triangle with 0, 1, or 2 segments, and we put $\mathrm{Ar}(\widetilde{\Delta}_k)$ to be the lattice area of that triangle.

Theorem 1.3 *In the notation of section 1.3.2, if* $\Sigma = \mathbb{S}^2_{0,2}$*, and the positive integers* r', r'' *satisfy (1.6), then*

$$W_{r''}(\Sigma, \mathcal{L}_\Delta) = \sum W(\mathcal{R}, \gamma, S, G, \overline{s}', \overline{s}'') \ , \tag{1.14}$$

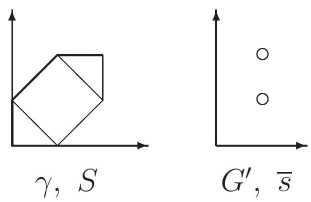


$$\begin{cases} \mathcal{R} = \{r' = 1+2,\ r'' = 0+1+0\} \\ \widetilde{m} = 1, \quad n = 2 \\ W = 4 \end{cases}$$

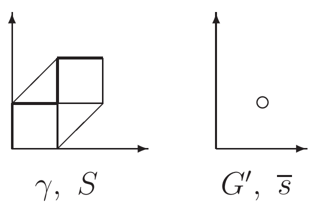


$$\begin{cases} \mathcal{R} = \{r' = 1+2,\ r'' = 0+0+1\} \\ \widetilde{m} = 1, \quad n = 1 \\ W = -2 \end{cases}$$

(a) Case $r' = 3,\ r'' = 1,\ W = 2$



$$\begin{cases} \mathcal{R} = \{r' = 0+1,\ r'' = 0+2+0\} \\ \widetilde{m} = 1, \quad n = 2 \\ W = 2 \end{cases}$$



$$\begin{cases} \mathcal{R} = \{r' = 0+1,\ r'' = 0+1+1\} \\ \widetilde{m} = 1, \quad n = 1 \\ W = -2 \end{cases}$$

(b) Case $r' = 1,\ r'' = 2,\ W = 0$

Figure 7: Admissible paths, graphs, and markings, IV

where the sum ranges over splittings $\mathcal{R}$ of r', r'', satisfying (1.8), all $\mathcal{R}$-admissible lattice paths γ, all γ-admissible subdivisions S of Δ, all $(\gamma, \mathcal{R}, S)$-admissible graphs G, and all markings $\overline{s}', \overline{s}''$ of G, subject to conditions, specified in section 1.3.2.

As example we consider the linear system $|\mathcal{L}_\Delta|$ with the hexagon Δ shown in Figure 1(d) for $d = 2$, $d_1 = 1$. Here $r' + 2r'' = 5$, and the integration with respect to Euler characteristic (cf. section 1.2.3, case (A)) gives $W_{r''}(\mathbb{S}^2_{0,2}, \mathcal{L}_\Delta) = 4 - 2r''$, $r'' = 1, 2$. In Figure 7 we demonstrate how to obtain these answers from formula (1.14), listing admissible paths γ, subdivisions S of Δ, and graphs G' (here the marked points are denoted by bullets and the non-marked one-point components of G' are denoted by circles).

2 Tropical limits of real rational curves on non-standard real toric Del Pezzo surfaces

2.1 Preliminaries

Here we recall definitions and a few facts about tropical curves and tropical limits of algebraic curves over a non-Archimedean field, presented in [6, 4, 13, 14, 16, 17, 18] in more details.

By Kapranov's theorem the amoeba A_C of a curve $C \in |\mathcal{L}_\Delta|_{\mathbb{K}}$, given by an equation

$$f(x,y) := \sum_{(i,j)\in\Delta} a_{ij}x^iy^j = 0, \quad a_{ij} \in \mathbb{K},\ (i,j) \in \Delta \cap \mathbb{Z}^2, \tag{2.15}$$

with the Newton polygon Δ, is the corner locus of the convex piece-wise linear function

$$N_f(x,y) = \max_{(i,j)\in\Delta\cap\mathbb{Z}^2}(xi + yj + \mathrm{Val}(a_{ij})), \quad x, y \in \mathbb{R}\ . \tag{2.16}$$

In particular, A_C is a planar graph with all vertices of valency ≥ 3.

Take the convex polyhedron

$$\widetilde{\Delta} = \{(i,j,\gamma) \in \mathbb{R}^3\ :\ \gamma \ge -\mathrm{Val}(a_{ij}), \quad (i,j) \in \Delta \cap \mathbb{Z}^2\}$$

and define the function

$$\nu_f : \Delta \to \mathbb{R}, \quad \nu_f(x,y) = \min\{\gamma\ :\ (x,y,\gamma) \in \widetilde{\Delta}\}\ . \tag{2.17}$$

This is a convex piece-wise linear function, whose linearity domains form a subdivision S_C of Δ into convex lattice polygons $\Delta_1, ..., \Delta_N$. The function ν_f is Legendre dual to N_f, and the subdivision S_C is combinatorially dual to the pair $(\mathbb{R}^2, A_C)$. Clearly, A_C and S_C do not depend on the choice of a polynomial f defining the curve C.

We define the *tropical curve*, corresponding to the algebraic curve C, as the weighted graph, supported at A_C, i.e., the non-Archimedean amoeba A_C, whose edges are assigned the weights equal to the lattice lengths of the dual edges of S_C. The subdivision S_C can be uniquely restored from the tropical curve A_C.

By the *tropical limit* of a curve C given by (2.15) we call a pair $(A_C, \{C_1, ..., C_N\})$, where C_k, $1 \le k \le N$, is a complex curve on the toric surface $\mathrm{Tor}(\Delta_k)$, associated with a polygon Δ_k from the subdivision S_C, and is defined by an equation

$$f_k(x,y) := \sum_{(i,j)\in\Delta_k} a^0_{ij}x^iy^j = 0\ ,$$

where $a_{ij}(t) = (a^0_{ij} + O(t^{>0})) \cdot t^{\nu_f(i,j)}$ is the coefficient of x^iy^j in $f(x,y)$. We call $C_1,...,C_N$ *limit curves*. Their geometrical meaning is as follows (cf. [17], section 2). By a parameter change $t \mapsto t^M$, $M >> 1$, we can make all the exponents of t in the coefficients $a_{ij} = a_{ij}(t)$ of f integral, and make the function ν_f integral-valued at integral points. The toric threefold $Y = \mathrm{Tor}(\widetilde{\Delta})$ fibers over $\mathbb{C}$ so that Y_t, $t \ne 0$, is isomorphic to $\mathrm{Tor}(\Delta)$, and Y_0 is the union of $\mathrm{Tor}(\Delta_k)$ attached to each other as the polygons of the subdivision S_C. Equation (2.15) defines an analytic surface C in Y such that the curves $C^{(t)} = C \cap Y_t$, $0 < |t| < \varepsilon$, form an equisingular family, and $C^{(0)} = C \cap Y_0 = C_1 \cup ... \cup C_N$, where $C_k = C \cap \mathrm{Tor}(\Delta_k)$.

Now let $\Sigma = \mathbb{S}^2$, $\mathbb{S}^2_{1,0}$, $\mathbb{S}^2_{2,0}$, or $\mathbb{S}^2_{0,2}$ be defined over the field $\mathbb{K}$, and let Δ be one of the respective polygons shown in Figure 1(a-d). Fix some non-negative integers r', r'' satisfying (0.1) and pick a configuration $\overline{\boldsymbol{p}}$ of $-c_1(\mathcal{L}_\Delta)K_\Sigma - 1$ distinct points in $(\mathbb{K}^*)^2 \subset \Sigma$ such that $\overline{\boldsymbol{p}} = \overline{\boldsymbol{p}}' \cup \overline{\boldsymbol{p}}''$ with

$$\begin{cases} \overline{\boldsymbol{p}}' = \{\boldsymbol{p}'_1, ..., \boldsymbol{p}'_{r'}\} \subset \Sigma(\mathbb{K}_\mathbb{R}), \\ \overline{\boldsymbol{p}}'' = \{\boldsymbol{p}''_{1,1}, \boldsymbol{p}''_{1,2}, ..., \boldsymbol{p}''_{r'',1}, \boldsymbol{p}''_{r'',2}\} \subset \Sigma(\mathbb{K})\backslash\Sigma(\mathbb{K}_R), \\ \mathrm{Conj}(\boldsymbol{p}''_{i,1}) = \boldsymbol{p}''_{i,2}, \quad i = 1, ..., r''. \end{cases} \tag{2.18}$$

Since the anti-holomorphic involution acts on $(\mathbb{K}^*)^2 \subset \Sigma(\mathbb{K})$ by $\mathrm{Conj}(\xi, \eta) = (\overline{\eta}, \overline{\xi})$, we have $\boldsymbol{p}'_i = (\xi_i(t), \overline{\xi}_i(t))$, $i = 1, ..., r'$, and $\boldsymbol{p}''_{i,1} = (\eta_i(t), \zeta_i(t))$, $\boldsymbol{p}''_{i,2} = (\overline{\zeta}_i(t), \overline{\eta}_i(t))$, $i = 1, ..., r''$. In particular, the configuration $\overline{\boldsymbol{x}}' = \mathrm{Val}(\overline{\boldsymbol{p}}')$ lies on $\mathcal{B}$, and the configuration $\overline{\boldsymbol{x}}'' = \mathrm{Val}(\overline{\boldsymbol{p}}'')$ is symmetric with respect to $\mathcal{B}$. We assume $\overline{\boldsymbol{p}}$ to be generic in $\Omega_{r''}(\Sigma(\mathbb{K}), \mathcal{L}_\Delta)$ and such that the configuration $\overline{\boldsymbol{x}} = \overline{\boldsymbol{x}}' \cup \overline{\boldsymbol{x}}''$ consists of $r' + r''$ generic distinct points on $\mathcal{B}$:

$$\overline{\boldsymbol{x}}' = \{\boldsymbol{x}'_i \ : \ \boldsymbol{x}'_i = \mathrm{Val}(\boldsymbol{p}'_i),\ i = 1, ..., r'\}\ ,$$

$$\overline{\boldsymbol{x}}'' = \{\boldsymbol{x}''_i \ : \ \boldsymbol{x}''_i = \mathrm{Val}(\overline{\boldsymbol{p}}''_{i,1}) = \mathrm{Val}(\boldsymbol{p}''_{i,2}),\ i = 1, ..., r''\}\ .$$

In addition, we require the following property:

($\boldsymbol{x}\mathbf{1}$) the points of $\overline{\boldsymbol{x}}$ are ordered on $\mathcal{B}$ as

$$\boldsymbol{x}''_1 \prec ... \prec \boldsymbol{x}''_{r''} \prec \boldsymbol{x}'_1 \prec ... \prec \boldsymbol{x}'_{r'}$$

by the growing sum of coordinates, and, moreover, the distance between any pair of neighboring points is much larger than that for the preceding pair.

Let $C \in |\mathcal{L}_\Delta|_\mathbb{K}$ be a real rational curve, passing through $\overline{\boldsymbol{p}}$. We can define C by an equation (2.15) with $a_{ji} = \overline{a}_{ij}$, $(i,j) \in \Delta$. Observe that the tropical curve $A_C \subset \mathbb{R}^2$ is symmetric with respect to $\mathcal{B}$, and so is the dual subdivision S_C of Δ. We intend to describe the tropical limits $(A_C, \{C_1, ..., C_N\})$ of such curves C.

2.2 Tropical limits of real rational curves on $\mathbb{S}^2$, $\mathbb{S}^2_{1,0}$, or $\mathbb{S}^2_{2,0}$

In addition to the notation of the preceding section, introduce the following ones:

- let $P(S_\Delta)$, $E(S_C)$, and $V(S_C)$ be the sets of the polygons, the edges, and the vertices of S_C;
- $P(S_C)$ splits into the disjoint subsets $P_{\mathcal{B}}(S_C)$, containing the polygons symmetric with respect to $\mathcal{B}$, and $P_+(S_C)$, $P_-(S_C)$, consisting of the polygons contained in the half-planes $\mathcal{B}_+$, $\mathcal{B}_-$, respectively; notice that the polygons of $P_+(S_C)$ and $P_-(S_C)$ are in 1-to-1 correspondence by the reflection with respect to $\mathcal{B}$, and that the limit curves C_i, $\Delta_i \in P_{\mathcal{B}}(S_C)$ are real, and the limit curves C_j, $\Delta_j \notin P_{\mathcal{B}}(S_C)$ are not;
- for a limit curve C_i, $1 \le i \le N$, denote by C_{ij}, $1 \le j \le l_i$, the set of all its components (counting each one with its multiplicity);
- for any $\Delta_k \in P(S_C)$, denote by $\mathrm{Tor}(\partial\Delta_k)$ the union of the toric divisors $\mathrm{Tor}(\sigma)$, $\sigma \subset \partial\Delta_k$.

Observe that all the curves C_{ij}, $1 \le j \le l_i$, $1 \le i \le N$, are rational (cf. [18], Step 1 of the proof of Proposition 2.1), which comes from the inequality for geometric genera $g(C^{(t)}) \ge \sum_{i,j} g(C_{ij})$ (see [3], Proposition 2.4, or [15]. We say that C_{ij} is **binomial** and write $C_{ij} \in \mathcal{C}_b$ if C_{ij} intersects with $\mathrm{Tor}(\partial\Delta_i)$ at precisely two points (in this case the two toric divisors, meeting C_{ij}, correspond to opposite parallel edges of Δ_i, and C_{ij} is defined by an irreducible binomial), otherwise we write $C_{ij} \in \mathcal{C}_{nb}$.

Now we split the configuration $\overline{\boldsymbol{x}}$ as follows:

$$\overline{\boldsymbol{x}}' = \overline{\boldsymbol{x}}'_1 \cup \overline{\boldsymbol{x}}'_2 \cup \overline{\boldsymbol{x}}'_3, \quad \overline{\boldsymbol{x}}'' = \overline{\boldsymbol{x}}''_1 \cup \overline{\boldsymbol{x}}''_2 \cup \overline{\boldsymbol{x}}''_3 \ ,$$

where

- $\overline{\boldsymbol{x}}'_1$ (resp., $\overline{\boldsymbol{x}}''_1$) consists of the points $\boldsymbol{x}'_i$, $1 \le i \le r'$ (resp., $\boldsymbol{x}''_i$, $1 \le i \le r''$), lying inside the edges of A_C on $\mathcal{B}$;
- $\overline{\boldsymbol{x}}'_2$ (resp., $\overline{\boldsymbol{x}}''_2$) consists of the points $\boldsymbol{x}'_i$, $1 \le i \le r'$ (resp., $\boldsymbol{x}''_i$, $1 \le i \le r''$), which are vertices of A_C;
- $\overline{\boldsymbol{x}}'_3$ (resp., $\overline{\boldsymbol{x}}''_3$) consists of the points $\boldsymbol{x}'_i$, $1 \le i \le r'$ (resp., $\boldsymbol{x}''_i$, $1 \le i \le r''$), lying inside edges of A_C orthogonal to $\mathcal{B}$.

Furthermore, a point $\boldsymbol{x}'_i \in \overline{\boldsymbol{x}}'_2$ (resp., $\boldsymbol{x}''_i \in \overline{\boldsymbol{x}}''_2$) is dual to a polygon $\Delta_k \in P_{\mathcal{B}}(S_C)$. In particular, the point $\boldsymbol{p}'_i \in (\mathbb{R}^*)^2 \subset \mathrm{Tor}(\Delta_k)$ (resp., the points $\boldsymbol{p}''_{i,1}, \boldsymbol{p}''_{i,2} \in (\mathbb{C}^*)^2 \subset$

$\mathrm{Tor}(\Delta_k)$) lies on the limit curve C_k. For a point $\boldsymbol{p} = (\xi(t),\eta(t)) \in (\mathbb{K}^*)^2$, we put $\mathrm{Ini}(\boldsymbol{p}) = (\xi_0,\eta_0) \in (\mathbb{C}^*)^2$, ξ_0,η_0 being the coefficients of the lower powers of t in $\xi(t),\eta(t)$, respectively. We then have $\overline{\boldsymbol{x}}'_2 = \overline{\boldsymbol{x}}'_{2,1} \cup \overline{\boldsymbol{x}}'_{2,2} \cup \overline{\boldsymbol{x}}'_{2,3} \cup \overline{\boldsymbol{x}}'_{2,4}$ and $\overline{\boldsymbol{x}}''_2 = \overline{\boldsymbol{x}}''_{2,1} \cup \overline{\boldsymbol{x}}''_{2,2} \cup \overline{\boldsymbol{x}}''_{2,3} \cup \overline{\boldsymbol{x}}''_{2,4}$, where

- $\overline{\boldsymbol{x}}'_{2,1}$ (resp., $\overline{\boldsymbol{x}}''_{2,1}$) consists of the points $\boldsymbol{x}'_i$, $1 \le i \le r'$ (resp., $\boldsymbol{x}''_i$, $1 \le i \le r''$), such that $\mathrm{Ini}(\boldsymbol{p}'_i)$ lies (resp., both $\mathrm{Ini}(\boldsymbol{p}''_{i,1})$ and $\mathrm{Ini}(\boldsymbol{p}''_{i,2})$ lie) on a real non-binomial component C_{kl} of C_k; we, furthermore, make splitting $\overline{\boldsymbol{x}}''_{2,1} = \overline{\boldsymbol{x}}''_{2,1a} \cup \overline{\boldsymbol{x}}''_{2,1b}$, where a point $\boldsymbol{x}''_i \in \overline{\boldsymbol{x}}''_{2,1}$, dual to $\Delta_k \in P_{\mathcal{B}}(S_C)$, belongs to $\overline{\boldsymbol{x}}''_{2,1a}$ or to $\overline{\boldsymbol{x}}''_{2,1b}$ according as C_{kl} has one or at least two local branches centered along $\mathrm{Tor}((\partial\Delta_k)_+)$;

- $\overline{\boldsymbol{x}}'_{2,2}$ (resp., $\overline{\boldsymbol{x}}''_{2,2}$) consists of the points $\boldsymbol{x}'_i$, $1 \le i \le r'$ (resp., $\boldsymbol{x}''_i$, $1 \le i \le r''$), such that the point $\mathrm{Ini}(\boldsymbol{p}'_i)$ lies (resp., the points $\mathrm{Ini}(\boldsymbol{p}''_{i,1})$, $\mathrm{Ini}(\boldsymbol{p}''_{i,2})$ lie) on two distinct conjugate components of C_k which cross $\mathrm{Tor}((\partial\Delta_k)_+)$;

- $\overline{\boldsymbol{x}}'_{2,3}$ (resp., $\overline{\boldsymbol{x}}''_{2,3}$) consists of the points $\boldsymbol{x}'_i$, $1 \le i \le r'$ (resp., $\boldsymbol{x}''_i$, $1 \le i \le r''$), such that $\mathrm{Ini}(\boldsymbol{p}'_i)$ lies on a real binomial component C_{kj} (resp., $\mathrm{Ini}(\boldsymbol{p}''_{i,1})$, $\mathrm{Ini}(\boldsymbol{p}''_{i,2})$ lie on two distinct conjugate binomial components G_{kj}, C_{kl}) crossing $\mathrm{Tor}((\partial\Delta_k)_\perp)$.

Put $r'_i = \#(\overline{\boldsymbol{x}}'_i)$, $r''_i = \#(\overline{\boldsymbol{x}}''_i)$, $i = 1,2,3$, $r'_{2,j} = \#(\overline{\boldsymbol{x}}'_{2,j})$, $r''_{2,j} = \#(\overline{\boldsymbol{x}}''_{2,j})$, $j = 1,2,3,4$, $r''_{2,1a} = \#(\overline{\boldsymbol{x}}''_{2,1a})$, $r''_{2,1b} = \#(\overline{\boldsymbol{x}}''_{2,1b})$.

Projecting the polygons $\Delta_k \in P_{\mathcal{B}}(S_C)$ to $(\partial\Delta)_+$ in the direction orthogonal to $\mathcal{B}$, we obtain

$$|\partial\Delta| - |(\partial\Delta)_\perp| = 2|(\partial\Delta)_+| = 2 \sum_{\substack{\sigma\subset(\partial\Delta_k)_+\\ \Delta_k\in P_{\mathcal{B}}(S_C)}} \mathrm{pr}(\sigma) + 4 \sum_{\substack{\sigma\in E(S_C)\\ \sigma\subset\mathcal{B}}} |\sigma|$$

$$\ge 2(r'_{2,1} + 2r'_{2,2} + r'_{2,3} + r''_{2,1a} + 2r''_{2,1b} + 2r''_{2,2} + r''_{2,3}) + 2(2r'_3 + 4r''_3)\ , \qquad (2.19)$$

where the equality holds only if

(E1) each polygon $\Delta_k \in P_{\mathcal{B}}(S_C)$ is dual to a point of $\overline{\boldsymbol{x}}'_2 \cup \overline{\boldsymbol{x}}''_2$, and

- either $\mathrm{pr}((\partial\Delta_k)_+) = |(\partial\Delta_k)_+| = 1$, the limit curve C_k may have binomial components, meeting the toric divisors $\mathrm{Tor}(\sigma)$, $\sigma \subset (\partial\Delta_k)_\perp$, and it has one real non-multiple component, crossing the toric divisor $\mathrm{Tor}((\partial\Delta_k)_+)$,

- or $\mathrm{pr}((\partial\Delta_k)_+) = |(\partial\Delta_k)_+| = 2$, the limit curve C_k may have binomial components, meeting the toric divisors $\mathrm{Tor}(\sigma)$, $\sigma \subset (\partial\Delta_k)_\perp$, and it has two distinct non-multiple conjugate components, meeting the toric divisors $\mathrm{Tor}((\partial\Delta_k)_+)$,

- or $\mathrm{pr}((\partial\Delta_k)_+) = |(\partial\Delta_k)_+| = 2$, the limit curve C_k may have binomial components, meeting the toric divisors $\mathrm{Tor}(\sigma)$, $\sigma \subset (\partial\Delta_k)_\perp$, and it has one real non-multiple component, crossing the toric divisors $\mathrm{Tor}((\partial\Delta_k)_+)$;

(**E2**) any point $\mathrm{Ini}(\boldsymbol{p}'_i)$ (resp., $\mathrm{Ini}(\boldsymbol{p}''_{i,1})$ or $\mathrm{Ini}(\boldsymbol{p}''_{i,2})$) with $\boldsymbol{x}'_i \in \overline{\boldsymbol{x}}'_2$ (resp., $\boldsymbol{x}''_i \in \overline{\boldsymbol{x}}''_2$), dual to a polygon $\Delta_k \in P_{\mathcal{B}}(S_C)$, lies precisely on one component of the limit curve C_k.

Next, inequality (2.7) from [18], in our situation, reads

$$\sum_{\Delta_k\in P_{\mathcal{B}}(S_C)}\sum_{j=1}^{l_k}(B(C_{kj})-2)+\sum_{\Delta_k\in P(S_C)\setminus P_{\mathcal{B}}(S_C)}\sum_{j=1}^{l_k}(B(C_{kj})-2)$$

$$\le B(\partial\Delta)-2\le|\partial\Delta|-2\ , \tag{2.20}$$

where $B(C_{kj})$ is the number of the local branches of the curve C_{kj} centered along $\mathrm{Tor}(\Delta_k)$, and $B(\partial\Delta)$ is the number of the local branches of all the curves C_k (counting multiplicities) centered along the toric divisors $\mathrm{Tor}(\sigma)$, $\sigma \subset \Delta_k \cap \partial\Delta$, $k = 1, ..., N$. The equalities in (2.20) hold only if

(**E3**) local branches of the curves C_k centered along $\mathrm{Tor}(\sigma)$, $\sigma \subset \Delta_k \cap \partial\Delta$, $1 \le k \le N$, are nonsingular and transverse to $\mathrm{Tor}(\sigma)$;

(**E4**) for any edge $\sigma = \Delta_k \cap \Delta_l$ and any point $p \in C_k \cap C_l \subset \mathrm{Tor}(\sigma)$, the numbers of local branches of $(C_k)_{\mathrm{red}}$ and $C_l)_{\mathrm{red}}$ at p coincide (cf. [17], Remark 3.4);

(**E5**) no singular point of any curve $(C_k)_{\mathrm{red}}$ in $(\mathbb{C}^*)^2 \subset \mathrm{Tor}(\Delta_k)$, $k = 1, ..., N$, is smoothed up in the deformation $C^{(t)}$, $t \in (\mathbb{C}, 0)$.

To proceed with estimations, we introduce some auxiliary objects. For any edge $\sigma \subset \bigcup_{\Delta_k\in P_{\mathcal{B}}(S_C)}(\partial\Delta_k)_\perp$, there is a canonical identification $\pi_\sigma : \mathbb{P}^1 \to \mathrm{Tor}(\sigma)$. For such an edge σ, put $\Phi_\sigma = \mathrm{Tor}(\sigma) \cap C^{(0)}$ and define $\Phi = \bigcup_\sigma \pi_\sigma^{-1}(\Phi_\sigma) \subset \mathbb{C}^*$. Denote by $\widetilde{\Phi}$ the set of the local branches of the non-binomial components of the curves $(C_k)_{\mathrm{red}}$, $\Delta_k \in P_{\mathcal{B}}(S_C)$, centered at $\bigcup_\sigma \Phi_\sigma$. We claim that

$$\#(\widetilde{\Phi}) \ge 2(r'_1 + 2r''_1 + r'_{2,3} + 2r''_{2,3} + r''_{2,1a}) - B_0$$

$$\ge 2(r'_1 + 2r''_1 + r'_{2,3} + 2r''_{2,3} + r''_{2,1a}) - |(\partial\Delta)_\perp|\ , \tag{2.21}$$

where

$$B_0 = \#\left(C^{(0)} \cap \mathrm{Tor}\left(\bigcup_{\Delta_k\in P_{\mathcal{B}}(S_C)}(\partial\Delta_k)_\perp \cap \partial\Delta\right)\right)$$

Indeed, observe that

- each point $\boldsymbol{p} \in \overline{\boldsymbol{p}}$ such that $\mathrm{Val}(\boldsymbol{p}) \in \overline{\boldsymbol{x}}'_1 \cup \overline{\boldsymbol{x}}''_1 \cup \overline{\boldsymbol{x}}'_{2,3} \cup \overline{\boldsymbol{x}}''_{2,3}$ defines a point in Φ, and by our choice all these points are distinct and generic;
- binomial components of the curves C_k, $\Delta_k \in P_{\mathcal{B}}(S_C)$, may join only the points in $\bigcup_\sigma \Phi_\sigma$ with the same image in Φ.

Furthermore, let $I \subset \{1, 2, ..., N\}$ consist of all k such that the polygon Δ_k belongs to $P_{\mathcal{B}}(S_C)$ and is dual to a point of $\overline{\boldsymbol{x}}''_{2,1a}$. Let C_k^{nb} be the non-binomial component of the limit curve C_k, where $k \in I$, and let Δ_k^{nb} be its Newton polygon. Since $|(\partial\Delta_k^{nb})_+| = |(\partial\Delta_k)_+| = 1$, the space of rational curves in the linear system $|\mathcal{L}_{\Delta_k^{nb}}|$ on the surface $\mathrm{Tor}(\Delta_k)$, passing through the points $\mathrm{Ini}(\boldsymbol{p}''_{i,1}), \mathrm{Ini}(\boldsymbol{p}''_{i,2})$, where $\boldsymbol{x}''_i \in \overline{\boldsymbol{x}}''_{2,1a}$ is dual to Δ_k, is of dimension $|(\partial\Delta_k^{nb})_\perp| - 1$. That is, if s is the total number of the local branches of all the curves C_k^{nb}, centered along $\mathrm{Tor}((\partial\Delta_k)_\perp)$, $k \in I$, no more than $s - \#(I)$ of them can be chosen in a generic position, and hence the bound (2.21) follows. Next, we derive

$$\sum_{\Delta_k \in P_{\mathcal{B}}(S_C)} \sum_{j=1}^{l_k} (B(C_{kj}) - 2) \ge \#(\widetilde{\Phi}) + 2r''_{2,1b}$$

$$\ge 2(r'_1 + 2r''_1 + r'_{2,3} + 2r''_{2,3} + r''_{2,1a} + r''_{2,1b}) - |(\partial\Delta)_\perp| \ , \tag{2.22}$$

where the equality holds only if

(**E6**) the sides of Δ, orthogonal to $\mathcal{B}$, are sides of some polygons $\Delta_k \in P_{\mathcal{B}}(S_C)$, and all the intersection points of $C^{(0)}$ with $\mathrm{Tor}((\partial\Delta)_\perp)$ are non-singular and transversal;

(**E7**) $\#(\Phi) = 2(r'_1 + 2r''_1 + r'_{2,3} + 2r''_{2,3} + r''_{2,1a})$, all the points of $\bigcup_\sigma \Phi_\sigma$ with the same image in Φ are joined by binomial components of the curves C_k, $\Delta_k \in P_{\mathcal{B}}(S_C)$, into one connected component, and any curve $(C_k)_{\mathrm{red}}$ with $\Delta_k \in P_{\mathcal{B}}(S_C)$ is unibranch at each point of Φ_σ, $\sigma \subset (\partial\Delta_k)_\perp$.

Now, from (1.6) we derive

$$2|\partial\Delta| - 2 = 2r' + 4r'' = 2(r'_1 + r'_{2,1} + r'_{2,2} + r'_{2,3} + r'_3)$$

$$+4(r''_1 + r''_{2,1a} + r''_{2,1b} + r''_{2,2} + r''_{2,3} + r''_3) \ , \tag{2.23}$$

which after subtracting inequality (2.19) gives

$$|\partial\Delta| + |(\partial\Delta)_\perp| - 2 \le 2r'_1 + 4r''_1 + 2r''_{2,1a} + 2r''_{2,3} - 2r'_{2,2} - 2r'_3 - 4r''_3 \ . \tag{2.24}$$

On the other hand, (2.20) and (2.22) yield

$$|\partial\Delta| + B_0 - 2 \ge \sum_{\Delta_k\in P(S_C)\backslash P_{\mathcal{B}}(S_C)} \sum_{j=1}^{l_k}(B(C_{kj}) - 2)$$

$$+2r'_1 + 4r''_1 + 2r'_{2,3} + 2r''_{2,1a} + 2r''_{2,1b} + 4r''_{2,3}\ ,$$

which together with (2.24) and $B_0 \le |(\partial\Delta)_\perp|$ results in

$$B(C_{kj}) = 2 \quad \text{for all} \quad \Delta_k \in P(S_C)\backslash P_{\mathcal{B}}(S_C),\ j = 1, ..., l_k\ , \tag{2.25}$$

$$r'_{2,3} = r''_{2,3} = r''_{2,1b} = r'_{2,2} = r'_3 = r''_3 = 0\ , \tag{2.26}$$

and implies the equalities in (2.19), (2.20), (2.22), that is all the conditions (**E1**)-(**E7**) hold true as well as

(**E8**) for any $\Delta_k \in P(S_C)\backslash P_{\mathcal{B}}(S_C)$, the curve C_k consists of only binomial components (see (2.25)).

Remark 2.1 *From conditions (***E1***)-(***E9***) and equalities (2.25), (2.26), one can easily derive that* $\bigcup_{\Delta_k\in P_{\mathcal{B}}(S_C)}(\partial\Delta_k)_+$ *with the respective vertices of the polygons* $\Delta_k \in P_{\mathcal{B}}(S_C)$ *forms a connected lattice path* γ *in* Δ*, satisfying conditions (*γ**1***)-(*γ**5***). Furthermore, due to (***E5***) and (***E8***),* γ *has no intersections with* $\mathcal{B}$ *besides its endpoints (since, otherwise, the curves* $C(t)$*,* $t \ne 0$*, would be reducible). At last, placing the configuration* $\overline{\boldsymbol{x}}$ *on* $\mathcal{B}$ *so that* $\overline{\boldsymbol{x}}''$ *will precede* $\overline{\boldsymbol{x}}'$*, we get (*$\gamma$**4***).*

2.3 Tropical limits of real rational curves on $\Sigma = \mathbb{S}^2_{0,2}$

In the considered situation, the argument of the preceding section leads to inequalities (2.20), (2.21), and (2.22), the latter one turning, due to $(\partial\Delta)_\perp = \emptyset$, $B_0 = 0$ (see Figure 1(d)), into

$$\sum_{\Delta_k\in P_{\mathcal{B}}(S_C)} \sum_{j=1}^{l_k}(B(C_{kj}) - 2) \ge 2(r'_1 + 2r''_1 + r'_{2,3} + 2r''_{2,3} + r''_{2,1a} + r''_{2,1b})\ , \tag{2.27}$$

Inequality (2.19) will be replaced by another relations for the parameter B_+, equal to the total number of the local branches of the curves $(C_k)_{\text{red}}$, $\Delta_k \in P_{\mathcal{B}}(S_C)$, centered on the divisors $\text{Tor}((\Delta_k)_+)$, and the local branches of the curves $(C_j)_{\text{red}}$, $\Delta_j \in P_+(S_C)$, centered on the divisors $\text{Tor}(\sigma)$, $\sigma \subset \mathcal{B}$.

We have two possibilities.

Case 1. Assume that

$$B_+ \le |(\partial\Delta)_+| = 2d - d_1 \ . \tag{2.28}$$

In the notation of section 2.2, this yields

$$|\partial\Delta| = 2|(\partial\Delta)_+| \ge 2(r'_{2,1}+2r'_{2,2}+r'_{2,3}+r''_{2,1a}+2r''_{2,1b}+2r''_{2,2}+r''_{2,3}+r'_3+2r''_3) \tag{2.29}$$

with an equality only if

(E1') for each polygon $\Delta_k \in P_{\mathcal{B}}(S_C)$,

- either $|(\partial\Delta_k)_+| = 1$, Δ_k is dual to a point from $\overline{\boldsymbol{x}}'_{2,1} \cup \overline{\boldsymbol{x}}''_{2,1a}$, the limit curve C_k may have binomial components, meeting the toric divisors $\mathrm{Tor}(\sigma)$, $\sigma \subset (\partial\Delta_k)_\perp$, and it has one real (possibly multiple) component, crossing the toric divisor $\mathrm{Tor}((\partial\Delta_k)_+)$,

- or $|(\partial\Delta_k)_+| = 2$, Δ_k is dual to a point from $\overline{\boldsymbol{x}}''_{2,2}$, the limit curve C_k may have binomial components, meeting the toric divisors $\mathrm{Tor}(\sigma)$, $\sigma \subset (\partial\Delta_k)_\perp$, and it has two distinct (possibly multiple) conjugate components, meeting the toric divisors $\mathrm{Tor}((\partial\Delta_k)_+)$;

(E2') any point $\mathrm{Ini}(\boldsymbol{p}'_i)$ (resp., $\mathrm{Ini}(\boldsymbol{p}''_{i,1})$ or $\mathrm{Ini}(\boldsymbol{p}''_{i,2})$) with $\boldsymbol{x}'_i \in \overline{\boldsymbol{x}}'_2$ (resp., $\boldsymbol{x}''_i \in \overline{\boldsymbol{x}}''_2$), dual to a polygon $\Delta_k \in P_{\mathcal{B}}(S_C)$, lies on precisely one (possibly multiple) component of the limit curve C_k.

Subtracting (2.29) from (2.23), we obtain

$$|\partial\Delta| - 2 \le 2r'_1 + 4r''_1 + 2r''_{2,1a} + 2r''_{2,3} - 2r'_{2,2} \ ,$$

whereas (2.20) and (2.27) lead to

$$|\partial\Delta| - 2 \ge \sum_{\Delta_k\in P(S_C)\backslash P_{\mathcal{B}}(S_C)}\sum_{j=1}^{l_k}(B(C_{kj})-2)$$

$$+2r'_1 + 4r''_1 + 2r'_{2,3} + 2r''_{2,1a} + 2r''_{2,1b} + 4r''_{2,3} \ . \tag{2.30}$$

Comparison of the two last inequalities results in (2.25) together with (2.26), where r'_3 and r''_3 are excluded, and also results in equalities in (2.20), (2.21), (2.27), (2.28), and (2.29), that is we obtain (**E3**)-(**E5**), (**E7**), and (**E8**), as well as $r''_{2,1b} = 0$ (cf. (2.26)). As in the end of section 2.2, observe that (**E5**) and (**E8**) imply, first, $r'_3 = r''_3 = 0$, and, second, that the number of the distinct local branches of the curves C_k, $1 \le k \le N$, centered along $\mathrm{Tor}(\sigma)$, $\sigma \subset \partial\Delta$, is equal to B_+, which in turn equals to $|\partial\Delta|$, and hence all these branches are not multiple. In particular, the above condition (**E2'**) can be replaced by (**E2**) from the preceding section, and (**E1'**) reduces to

(**E1"**) for each polygon $\Delta_k \in P_{\mathcal{B}}(S_C)$,

- either $|(\partial\Delta_k)_+| = 1$, Δ_k is dual to a point from $\overline{\boldsymbol{x}}'_{2,1} \cup \overline{\boldsymbol{x}}''_{2,1a}$, the limit curve C_k may have binomial components, meeting the toric divisors $\mathrm{Tor}(\sigma)$, $\sigma \subset (\partial\Delta_k)_\perp$, and it has one real non-multiple component, crossing the toric divisor $\mathrm{Tor}((\partial\Delta_k)_+)$,

- or $|(\partial\Delta_k)_+| = 2$, Δ_k is dual to a point from $\overline{\boldsymbol{x}}''_{2,2}$, the limit curve C_k may have binomial components, meeting the toric divisors $\mathrm{Tor}(\sigma)$, $\sigma \subset (\partial\Delta_k)_\perp$, and it has two distinct non-multiple conjugate components, meeting the toric divisors $\mathrm{Tor}((\partial\Delta_k)_+)$.

Remark 2.2 *In the same way as in Remark 2.1 we decide that* $\bigcup_{\Delta_k\in P_{\mathcal{B}}(S_C)}(\partial\Delta_k)_+$ *forms an admissible lattice path* γ *satisfying conditions* (γ**1**)*-*(γ**4**)*.*

Case 2. Assume that

$$B_+ = 2d - d_1 + n_0 = |(\partial\Delta)_+| + n_0, \quad n_0 > 0\ . \tag{2.31}$$

Step 1. Similarly to Case 1, we have

$$|\partial\Delta| + 2n_0 = 2|(\partial\Delta)_+| + 2n_0$$

$$\geq 2(r'_{2,1} + 2r'_{2,2} + r'_{2,3} + r''_{2,1a} + 2r''_{2,1b} + 2r''_{2,2} + r''_{2,3} + r'_3 + 2r''_3)\ , \tag{2.32}$$

with an equality only if conditions (**E1'**), (**E2'**) from Case 1 hold true.

Also we refine inequality (2.20) up to the following one. For an edge $\sigma = \Delta_k \cap \Delta_l$ with some $\Delta_k, \Delta_l \in P(S_C)$, denote by $d(\sigma)$ the absolute value of the difference between the number of local branches of $(C_k)_{\mathrm{red}}$, centered on $\mathrm{Tor}(\sigma)$, and the corresponding number for $(C_l)_{\mathrm{red}}$. Then, due to [17], Remark 3.4, inequality (2.20) refines up to

$$\sum_{\Delta_k\in P_{\mathcal{B}}(S_C)}\sum_{j=1}^{l_k}(B(C_{kj})-2) + \sum_{\Delta_k\in P(S_C)\backslash P_{\mathcal{B}}(S_C)}\sum_{j=1}^{l_k}(B(C_{kj})-2) + \sum_{\substack{\sigma\in E(S_C)\\ \sigma\not\subset\partial\Delta}} d(\sigma)$$

$$\leq B(\partial\Delta) - 2 \leq |\partial\Delta| - 2\ . \tag{2.33}$$

Step 2. The key ingredient in our argument is the relation

$$\sum_{\Delta_k\in P_+(S_C)}\sum_{j=1}^{l_k}(B(C_{kj})-2) + \sum_{\substack{\sigma\in E(S_C)\\ \sigma\subset\mathcal{B}_+,\ \sigma\not\subset\partial\Delta}} d(\sigma) \geq n_0\ . \tag{2.34}$$

To derive it, choose a generic vector $\overline{v} \in \mathbb{R}^2$, close to $(-1,1)$, and denote by $d_{\overline{v}}(\Delta_k, C_k)$ the difference between the number of the local branches of $(C_k)_{\rm red}$, centered on the divisors $\mathrm{Tor}(\sigma)$ for the sides σ of Δ_k, through which $\overline{v}$ enters Δ_k, and the number of the local branches of $(C_k)_{\rm red}$, centered on the divisors $\mathrm{Tor}(\sigma)$ for the sides σ of Δ_k, through which $\overline{v}$ leaves Δ_k. Clearly, for any $\Delta_k \in P(S_C)\backslash P_{\mathcal{B}}(S_C)$,

$$\sum_{j=1}^{l_k}(B(C_{kj})-2) \ge |d_{\overline{v}}(\Delta_k, C_k)| \ , \tag{2.35}$$

and the equality here holds only when non-binomial components of C_k are non-multiple. Summing up inequalities (2.35) over all $\Delta_k \in P(S_C)$, $\Delta_k \subset \mathcal{B}_+$, and using (2.31), we get (2.34).

Step 3. Subtracting (2.32) from (2.23), we obtain

$$|\partial\Delta| - 2 \le 2n_0 + 2r'_1 + 4r''_1 + 2r''_{2,1a} + 2r''_{2,3} - 2r'_{2,2} \ ,$$

whereas (2.27) and (2.33) yield

$$|\partial\Delta| - 2 \ge B(\partial\Delta) - 2 \ge \sum_{\Delta_k \in P(S_C)\backslash P_{\mathcal{B}}(S_C)} \sum_{j=1}^{l_k}(B(C_{kj})-2) + \sum_{\substack{\sigma \in E(S_C) \\ \sigma \not\subset \partial\Delta}} d(\sigma)$$

$$+2r'_1 + 4r''_1 + 2r'_{2,3} + 2r''_{2,1a} + 2r''_{2,1b} + 4r''_{2,3} \ .$$

Together with (2.33) and (2.34) this implies equalities in (2.32), (2.33), and (2.34) as well as the the relations

$$r'_{2,2} = r'_{2,3} = r''_{2,3} = r''_{2,1b} = 0, \quad B(\partial\Delta) = |\partial\Delta| \ , \tag{2.36}$$

and

$$\sum_{j=1}^{l_k}(B(C_{kj})-2) = d_{\overline{v}}(\Delta_k, C_k) = 0 \text{ for all } \Delta_k \subset \mathcal{B}_+ \ . \tag{2.37}$$

Also, for any polygon $\Delta_k \subset \mathcal{B}_+$, we get the absence of sides, perpendicular to $\mathcal{B}$ in such polygons (otherwise, reflecting $\overline{v}$ with respect to the normal to $\mathcal{B}$, we will break at least one relation (2.37)). In turn, the equality conditions provide us with the properties (**E1'**), (**E2**)-(**E5**), (**E7**), and the following one:

(**E13**) for any polygon Δ_k with a side $\sigma \subset \mathcal{B}$, the curve C_k has at most two distinct local branches centered along $\mathrm{Tor}(\sigma)$.

Furthermore, relation (2.31) and the equalities in (2.34), (2.35) yield that

$$d(\sigma) = 0 \quad \text{for all} \quad \sigma \in E(S_C),\ \sigma \not\subset \partial\Delta \ , \tag{2.38}$$

in particular,

$$\sum_{\Delta_k\subset\mathcal{B}_+}\sum_{j=1}^{l_k}(B(C_{kj})-2)=n_0\ . \tag{2.39}$$

The last equality in (2.36), the equalities in (2.35), and (2.39) tell us that

(**E14**) all the local branches of the curves C_k, $1\le k\le N$, centered along the divisors $\mathrm{Tor}(\sigma)$, $\sigma\subset\partial\Delta$, are non-multiple, non-singular, and transverse to $\mathrm{Tor}(\sigma)$.

Then, in particular, the polygons $\Delta_k\in P_{\mathcal{B}}(S_C)$ with $(\partial\Delta_k)_+\subset\partial\Delta$ possess the property (**E1"**), introduced in Case 1.

Step 4. We shall describe the polygons $\Delta_k\in P(S_C)\backslash P_{\mathcal{B}}(S_C)$ and the respective limit curves C_k.

Introduce a partial order in $P_+(S_C)$, $\Delta_j\subset\mathcal{B}_+$, saying that $\Delta_j\prec\Delta_l$ if $\Delta_j\cap\Delta_l=\sigma$ is a common side, through which $\overline{v}$ goes from Δ_j to Δ_l. Extend this partial order up to a complete one.

Let $\Delta_k\in P_+(S_C)$ be the first polygon with respect to the order defined. Notice that Δ_k has precisely two sides, through which $\overline{v}$ enters Δ_k. Indeed, in case there is only one such side we immediately obtain that $d_{\overline{v}}(\Delta_k,C_k)<0$. In case of more than two such sides, we obtain that more than two edges of the tropical curve A_C, starting at some points of $\overline{\boldsymbol{x}}$ on $\mathcal{B}$, merge at the vertex, dual to Δ_k, which would impose a restriction to the configuration $\overline{\boldsymbol{x}}$ in contradiction to the generality condition ($\boldsymbol{x}$**1**) from section 2.1. Let $\sigma^{(1)},\sigma^{(2)}$ be the sides of Δ_k, through which $\overline{v}$ enters Δ_k. Relations (2.37), (2.38), and properties (**E1'**), (**E13**) leave the only following options for Δ_k and C_k (see Figure 8, where colored lines designate the components of $(C_k)_{\mathrm{red}}$ and their intersection with the corresponding toric divisors):

- Δ_k is a parallelogram, C_k splits into binomial components - Figure 8(a);
- Δ_k is a triangle, $(C_k)_{\mathrm{red}}$ is irreducible and has only one local branch, centered along $\mathrm{Tor}(\sigma)$, for any side σ of Δ_k - Figure 8(b);
- Δ_k is a triangle, $(C_k)_{\mathrm{red}}$ splits into two components, these components intersect only in $(\mathbb{C}^*)^2\subset\mathrm{Tor}(\Delta_k)$, and each of them has only one local branch, centered along $\mathrm{Tor}(\sigma)$, for any side σ of Δ_k - Figure 8(c);
- Δ_k is a trapeze with a side $\sigma^{(3)}$, parallel to $\sigma^{(1)}$ (or $\sigma^{(2)}$), the curve $(C_k)_{\mathrm{red}}$ splits into a binomial component, crossing the divisors $\mathrm{Tor}(\sigma^{(3)})$ and $\mathrm{Tor}(\sigma^{(1)})$ (resp., $\mathrm{Tor}(\sigma^{(2)})$), and a non-binomial components, crossing the divisors $\mathrm{Tor}(\sigma)$, $\sigma\subset\partial\Delta_k$, $\sigma\ne\sigma^{(3)}$, and having only one local branch along each of these divisors - Figure 8(d) (resp., (e));

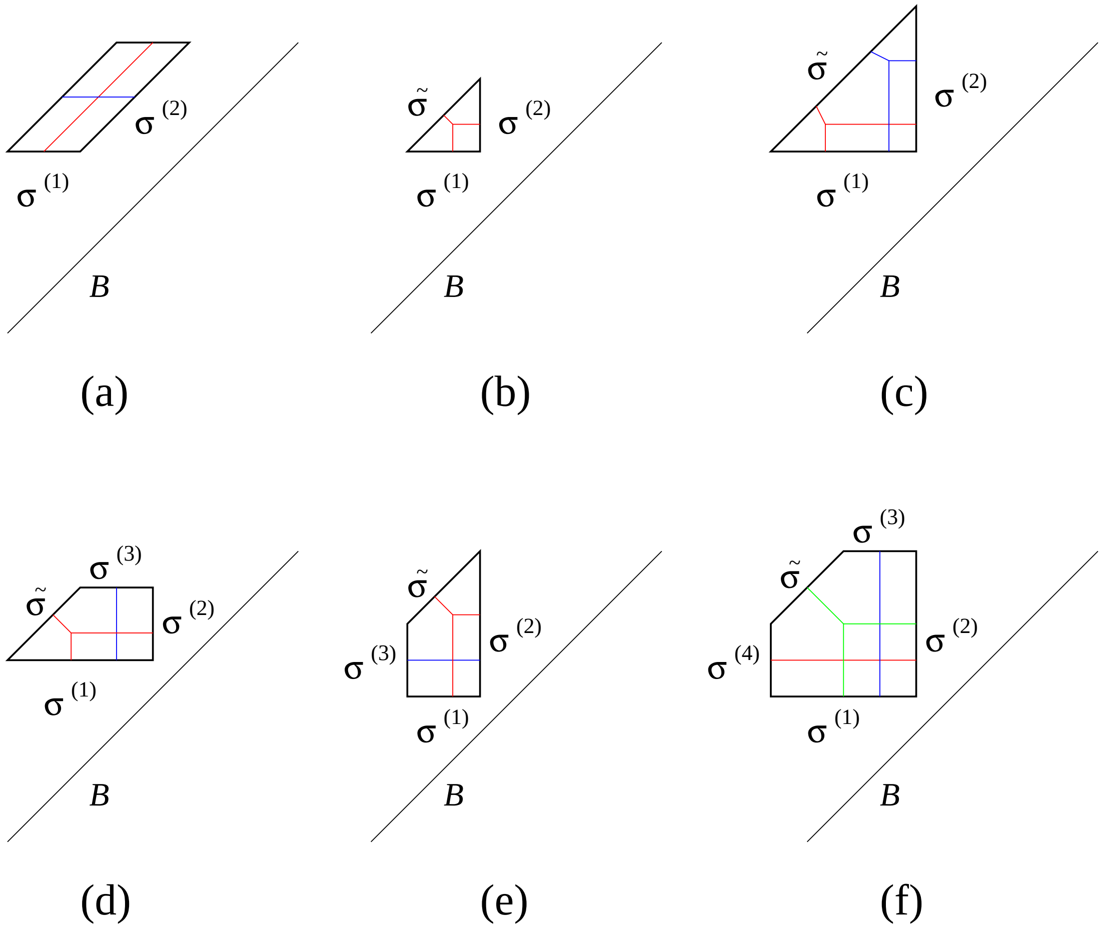


Figure 8: Polygons in $\mathcal{B}_+$

- Δ_k is a pentagon with sides $\sigma^{(3)}$, $\sigma^{(4)}$, parallel to $\sigma^{(1)}$, $\sigma^{(2)}$, respectively, the curve $(C_k)_{\rm red}$ splits into a binomial component, crossing the divisors $\text{Tor}(\sigma^{(1)}), \text{Tor}(\sigma^{(3)})$, a binomial component, crossing the divisors $\text{Tor}(\sigma^{(2)}), \text{Tor}(\sigma^{(4)})$, and a non-binomial component, crossing the divisors $\text{Tor}(\sigma)$, $\sigma \subset \partial\Delta_k$, $\sigma \ne \sigma^{(3)}, \sigma^{(4)}$, and having only one local branch along each of these divisors - Figure 8(f).

Now take the second polygon $\Delta_j \subset \mathcal{B}_+$ and observe that again there are precisely two sides $\sigma^{(1)}, \sigma^{(2)}$ of Δ_j, through which $\overline{v}$ enters Δ_j, and along each of the $\text{Tor}(\sigma^{(1)}), \text{Tor}(\sigma^{(2)})$, $(C_j)_{\rm red}$ has one or two local branches. Thus, we decide that Δ_j and C_j are as shown in Figure 8. Inductively we deduce the similar conclusions for all $\Delta_j \in P_+(S_C)$. Moreover, using condition (**E14**), obtained in Step 3, we can easily derive that

(**E15**) all $\Delta_k \in P_{\mathcal{B}}(S_C)$ possess property (**E1'**); all the limit curves C_k, $\Delta_k \in P(S_C) \backslash P_{\mathcal{B}}(S_C)$, are reduced, they cross $\text{Tor}(\partial\Delta_k)$ transversally at their non-

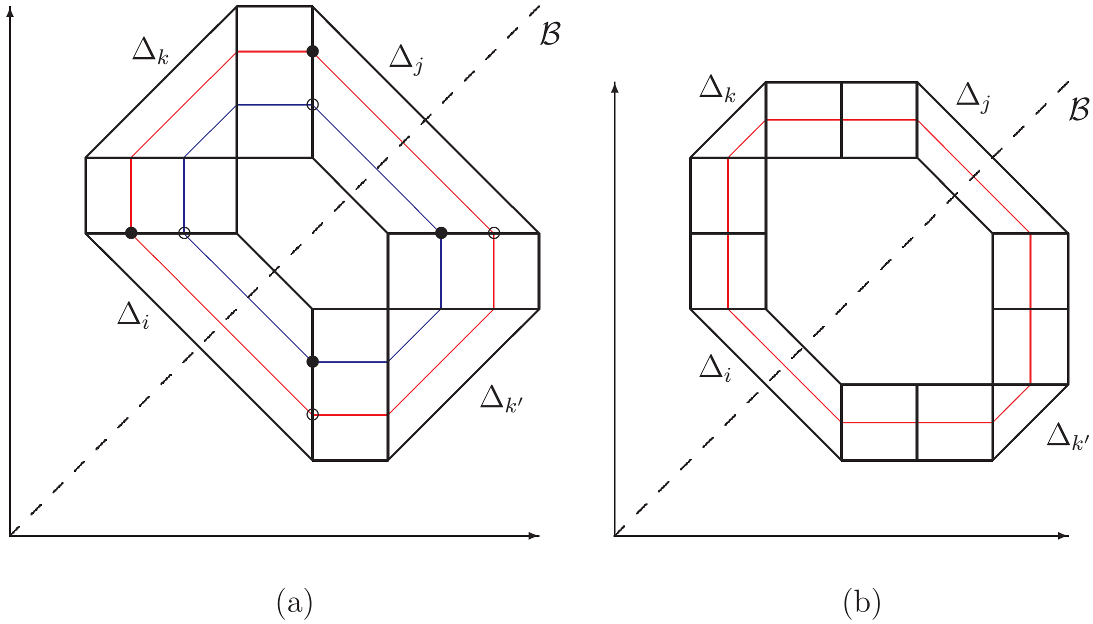


Figure 9: Forbidden subdivisions

singular points; if $\Delta_k \in P_+(S_C)$ is as shown in Figure 8(b-f), then the side $\widetilde{\sigma}$ is parallel to $\mathcal{B}$.

Taking into account condition (**E5**), we additionally obtain that $r_3' = r_3'' = 0$, and that $E(S_C)$ has no edges lying on $\mathcal{B}$.

Step 5. We claim that the triangles as shown in Figure 8(c) do not occur in S. Indeed, otherwise, a triangle Δ_k of this type and its symmetric with respect to $\mathcal{B}$ image $\Delta_{k'}$ would be joined by sequences of parallelograms with two trapezes $\Delta_i, \Delta_j \in P_{\mathcal{B}}(S_C)$, having non-parallel sides of length 2 and such that each of the curves C_i, C_j has two conjugate components, crossing $\mathrm{Tor}((\partial\Delta_i)_+), \mathrm{Tor}((\partial\Delta_j)_+)$, respectively, but then the components of the curves $C_k, C_{k'}$ and C_i, C_j (together with possible binomial components of the limit curves corresponding to parallelograms) would glue up in the deformation $C^{(t)}$, $t \in (\mathbb{C}, 0)$, so that the curves $C^{(t)}$, $t \neq 0$, would have at least two handles in contradiction to their rationality (see Figure 9(a), where the gluing components of the limit curves are designated by red and blue lines, the bullets and circles designate pairs of conjugate points of the limit curves on the toric divisors of $\mathrm{Tor}(\Delta_i)$ and $\mathrm{Tor}(\Delta_j)$).

Denoting by $n_{0,b}, n_{0,d}, n_{0,e}, n_{0,f}$ the number of the polygons in $P_+(S_C)$ of the types, shown in Figure 8(b,d-f), respectively, we derive from (2.31) and (**E15**) that

$$n_{0,b} + n_{0,d} + n_{0,e} + n_{0,f} = n_0 \ . \tag{2.40}$$

We also observe that the lattice path γ, defined as $\bigcup_{\Delta_k \in P_{\mathcal{B}}(S_C)} (\partial\Delta_k)_+$ with the common vertices of the polygons $\Delta_k \in P_{\mathcal{B}}(S_C)$ as the points v_i, $1 \le i < \widetilde{m}$, satisfies

the conditions (γ**1**)-(γ**5**) and (γ**1'**)-(γ**6'**) of sections 1.3.1 and 1.3.2. For example, condition (γ**6'**) holds, since, otherwise, one would have a polygon $\Delta_k \in P_{\mathcal{B}}(S_C)$ with $(\partial\Delta_k)_+$ consisting of of a segment with the projection 1/2 to $\mathcal{B}$ and of a unit segment parallel to $\mathcal{B}$, but such a polygon cannot split into the Minkowski sum of two polygons symmetric with respect to $\mathcal{B}$, contrary to the fact that the corresponding limit curve C_k contains two conjugate components, crossing $\text{Tor}((\partial\Delta_k)_+)$.

Remark 2.3 *Notice that the condition $r'' = 0$ (i.e., if all the fixed points are real) fits to the Case 1, defined by (2.28). Indeed, assuming in contrary (2.31) and taking into account the above conclusions (2.36), (***E1'***), $r'_3 = 0$, and (2.40), we derive that $(\partial\Delta_i)_+$ is a unit segment for any $\Delta_i \in P_{\mathcal{B}}(S_C)$, and thus, $P_+(S_C)$ contains n_0 triangles like shown in Figure 8(b). Then such a triangle Δ_k and its symmetric copy $\Delta_{k'}$ are joined with two polygons $\Delta_i, \Delta_j \in P_{\mathcal{B}}(S_C)$ by sequences of parallelograms (see, for instance, Figure 9(b)), and hence the corresponding limit curves (designated by red lines in Figure 9(b)) glue up in the deformation $C^{(t)}$, $t \in (\mathbb{C}, 0)$, into a non-rational curve $C^{(t)}$, $t \neq 0$, contradicting the initial assumptions.*

3 Proof of Theorems 1.1, 1.2, and 1.3

3.1 Encoding the tropical limits

Let us be given a surface $\Sigma = \mathbb{S}^2$, $\mathbb{S}^2_{1,0}$, $\mathbb{S}^2_{2,0}$, or $\mathbb{S}^2_{0,2}$, a respective lattice polygon Δ as shown in Figure 1(a-d), and a pair of non-negative integers $r'r''$, satisfying (1.6). Choose a generic configuration of points $\overline{\boldsymbol{p}} = \overline{\boldsymbol{p}}' \cup \overline{\boldsymbol{p}}'' \subset \Sigma(\mathbb{K})$ satisfying (2.18) and such that its valuation projection $\overline{\boldsymbol{x}} = \overline{\boldsymbol{x}}' \cup \overline{\boldsymbol{x}}'' \subset \mathcal{B}$ satisfies condition ($\boldsymbol{x}$**1**) from section 2.1.

For real rational curves $C \in |\mathcal{L}_\Delta|$ on the surface $\Sigma(\mathbb{K})$, passing through the configuration $\overline{\boldsymbol{p}}$, we have described possible tropical limits $(A_C, \{C_1, ..., C_N\})$. We encode them by means of the objects counted in Theorems 1.1, 1.2, and 1.3, and it is immediate from the results of section 2 that they satisfy all the conditions specified in section 1. Namely,

- the splitting $\mathcal{R}$ of r', r'' as in (1.7) is defined by taking r'_2 equal to the number of the points $\boldsymbol{x}'_i \in \overline{\boldsymbol{x}}'$ among the vertices of A_C, and taking $r''_{2,1}$ (resp., $r''_{2,2}$) equal to the numbers of the points $\boldsymbol{x}''_i \in \overline{\boldsymbol{x}}''$ among the vertices of A_C such that, for the polygon Δ_k in the dual subdivision S_C of Δ, it holds $|(\partial\Delta_k)_+| = 1$ (resp., $|(\partial\Delta_k)_+| = 2$);
- the broken line $\bigcup_{\Delta_k \in P_{\mathcal{B}}(S_C)} (\partial\Delta_k)_+$ naturally defines an admissible path γ with

the integral points on it v_i, $i = 0, ..., \widetilde{m}$, to be the endpoints of the fragments $(\partial\Delta_k)_+$, $\Delta_k \in P_{\mathcal{B}}(S_C)$;

- the intersection points of the curves C_k such that $\Delta_k \in P_{\mathcal{B}}(S_C)$ with the divisors $\mathrm{Tor}((\partial\Delta_k)_\perp)$, form the set of the vertices of the graph G', whereas the binomial components of these curves C_k, crossing $\mathrm{Tor}((\partial\Delta_k)_\perp)$, serve as the arcs of G' with their multiplicities as weights $w(G'_i)$; in turn, picking the vertices of G', which are $\mathrm{Ini}(\boldsymbol{p}'_i)$ or $\mathrm{Ini}(\boldsymbol{p}''_{i,1}), \mathrm{Ini}(\boldsymbol{p}_{i,2})$, we obtain the marking $\overline{s} = \overline{s}' \cup \overline{s}''$;

- then we take the components of the curves C_k, $\Delta_k \in P_{\mathcal{B}}(S)_C)$, which are not binomial crossing $\mathrm{Tor}((\partial\Delta_k)_\perp$, as additional vertices of the graph G, joining them by arcs with the vertices of G', which belong to these components;

- finally, we take the the non-binomial components of the curves C_j, $\Delta_j \in P(S_C)\backslash P_{\mathcal{B}}(S_C)$, and join them with the previously defined vertices of G as far as the corresponding components of the limit curves either intersect, or can be connected by a sequence of binomial components.

Remark 3.1 *Up to some details the graph G can be viewed as a rational parameterization of the tropical curve A_C.*

3.2 Restoring the tropical limits

Let a surface Σ, a polygon Δ, a pair of nonnegative integers r', r'', and a configuration $\overline{\boldsymbol{p}}$ as in the preceding section, and let $\mathcal{R}, \gamma, S, G, \overline{s}$ be suitable objects from Theorems 1.1, 1.2, or 1.3. We shall describe how to recover the tropical limits of the real rational curves on $\Sigma(\mathbb{K})$, compatible with the given data.

The subdivision S determines the combinatorial type of the tropical curve A, and to restore A completely we should pick r'_2 points from $\overline{\boldsymbol{x}}'$ and $r''_{2,1} + r''_{2,2}$ points from $\overline{\boldsymbol{x}}''$ and appoint them as vertices of A of the line $\mathcal{B}$. This choice of $r'_2 + r''_{2,1} + r''_{2,2} = \widetilde{m}$ points of $\overline{\boldsymbol{x}}$ is uniquely determined by the marking $\overline{s}$: namely, any $\widetilde{m}$ points of $\overline{\boldsymbol{x}}$ divide $\mathcal{B}$ into $\widetilde{m}+1$ naturally ordered intervals, and the distribution of the remaining points of $\overline{\boldsymbol{x}}'$ and $\overline{\boldsymbol{x}}''$ in these intervals must coincide with the distribution of the values of $\overline{s}'$ and $\overline{s}''$, respectively, in the intervals

$$\left(-\infty, \frac{1}{2}\right), \ \left(\frac{1}{2}, \frac{3}{2}\right), \ \ldots \ \left(\widetilde{m} - \frac{3}{2}, \widetilde{m} - \frac{1}{2}\right), \ \left(\widetilde{m} - \frac{1}{2}, \infty\right) \ .$$

The tropical curve A and the subdivision S determine a convex piece-wise linear function $\nu : \Delta \to \mathbb{R}$ uniquely up to a constant summand.

Now we pass to restoring limit curves $C_1, ..., C_N$.

First, notice that, knowing the limit curves C_k for all $\Delta_k \in P_{\mathcal{B}}(S)$, we uniquely define the remaining limit curves, provided that they consist of only binomial components. In case $\Sigma = \mathbb{S}^2_{0,2}$, $r'' > 0$, and $n_0 > 0$, the limit curves C_j with $\Delta_j \subset \mathcal{B}_+$ contain in total n_0 non-binomial components, which can be restored in $\prod_{k=1}^{n_0} \mathrm{Ar}(\widetilde{\Delta}_k)$ ways (see the notation of section 1.3.2) due to [17], Lemma 3.5. For $\Delta_j \subset \mathcal{B}_-$ we respectively take conjugate limit curves.

Any point $\boldsymbol{x}'_i$ (resp., $\boldsymbol{x}''_i$), lying inside an edge of A on $\mathcal{B}$, defines a real point $\mathrm{Ini}(\boldsymbol{p}'_i)$ (resp., a pair of conjugate points $\mathrm{Ini}(\boldsymbol{p}''_{i,1}), \mathrm{Ini}(\boldsymbol{p}''_{i,2})$) in $\mathrm{Tor}(\sigma)$ for the dual edge $\sigma \in E(S)$. Using the weighted graph G', we can restore the respective binomial components of the curves C_k, $\Delta_k \in P_{\mathcal{B}}(S)$, and this can be done in $\prod_{i=0}^{\widetilde{m}}(n'_i! n''_i! \alpha_i^{-1}\beta_i^{-1})$ ways (in the notation of section 2). Furthermore, for a curve C_k, where $\Delta_k \in P_{\mathcal{B}}(S)$ is dual to a point $\boldsymbol{x}''_i$, its binomial component, which crosses $\mathrm{Tor}((\partial\Delta_k)_+)$, must go through $\mathrm{Ini}(\boldsymbol{p}''_{i,1}$ or $\mathrm{Ini}(\boldsymbol{p}''_{i,2})$, and hence is defined uniquely.

To restore the remaining components of C_k, $\Delta_k \in P_{\mathcal{B}}(S)$, and count how many solutions are there, we use

Lemma 3.2 *Let a lattice polygon Δ_0 be a triangle with a side $\sigma' \perp \mathcal{B}$ and an opposite vertex σ'', or a trapeze with sides $\sigma', \sigma'' \perp \mathcal{B}$. Let us be given generic distinct points $z'_i \in \mathrm{Tor}(\sigma')$, $1 \le i \le p$ ($p \ge 0$), and, in case $|\sigma''| \ne 0$, generic distinct points $z''_j \in \mathrm{Tor}(\sigma'')$, $1 \le j \le q$ ($q \ge 0$), and let m'_i, $1 \le i \le p$, and m''_j, $1 \le j \le q$, be positive integers.*

(1) Assume that

$$m'_1 + ... + m'_p = |\sigma'|, \quad m''_1 + ... + m''_q = |\sigma''| \ , \tag{3.41}$$

$|\mathrm{pr}_{\mathcal{B}}(\Delta_0)| = 1/2$, *and* $z_0 \in (\mathbb{C}^*)^2 \subset \mathrm{Tor}(\Delta_0)$ *is a generic point. Then there is a unique rational curve $C_0 \in |\mathcal{L}(\Delta_0)|$ on the surface $\mathrm{Tor}(\Delta_0)$, passing through z_0 and satisfying*

$$C_0 \cap \mathrm{Tor}(\sigma') = \sum_{i=1}^{p} m'_i z'_i, \quad C_0 \cap \mathrm{Tor}(\sigma'') = \sum_{j=1}^{q} m''_j z''_j \ . \tag{3.42}$$

Furthermore, it is non-singular.

(2) Let Δ_0 be symmetric with respect to $\mathcal{B}$, the real structure on the surface $\mathrm{Tor}(\Delta_0)$ be defined by $\mathrm{Conj}(x,y) = (\overline{y}, \overline{x})$, $(x,y) \in (\mathbb{C}^*)^2$, and the divisors $\sum_i m'_i z'_i \subset \mathrm{Tor}(\sigma')$, $\sum_j m''_j z''_j \subset \mathrm{Tor}(\sigma'')$ be conjugation invariant.

- *(2i) Assume that Δ_0 is a rectangle with a unit length side parallel to $\mathcal{B}$, relation (3.41) holds true, $z_0 \in (\mathbb{R}^*)^2 \subset \mathrm{Tor}(\Delta_0)$ is a generic real point, and C_0 is unibranch at each point of $C_0 \cap (\mathrm{Tor}(\sigma') \cup \mathrm{Tor}(\sigma''))$. Then there are 2^{p+q}*

distinct real rational curves $C_0 \in |\mathcal{L}(\Delta_0)|$ on the surface $\mathrm{Tor}(\Delta_0)$*, passing through z_0 and satisfying (3.42). Moreover, they all are non-singular along* $\mathrm{Tor}(\sigma')$ *and* $\mathrm{Tor}(\sigma'')$*.*

(2ii) Assume that

$$m'_1 + ... + m'_p = |\sigma'| - (2l+1),\ l \ge 0, \quad m''_1 + ... + m''_q = |\sigma''| \ , \tag{3.43}$$

and $z_1, z_2 \in (\mathbb{C}^*)^2 \subset \mathrm{Tor}(\Delta_0)$ are distinct conjugate generic points. Then the number of real rational curves $C_0 \in |\mathcal{L}(\Delta_0)|$ on the surface $\mathrm{Tor}(\Delta_0)$*, passing through z_1, z_2, satisfying*

$$C_0 \cap \mathrm{Tor}(\sigma') = \sum_{i=1}^{p} m'_i z'_i + (2l+1) z'_{p+1}, \quad C_0 \cap \mathrm{Tor}(\sigma'') = \sum_{j=1}^{q} m''_j z''_j \tag{3.44}$$

with some $z'_{p+1} \in \mathrm{Tor}(\sigma')$, and unibranch at each point of $C_0 \cap (\mathrm{Tor}(\sigma') \cup \mathrm{Tor}(\sigma''))$, is equal to $2l+1$ if $|\mathrm{pr}_{\mathcal{B}}(\Delta_0)| = 1/2$, and is equal to 2^{p+q} if Δ_0 is a rectangle with a unit length side parallel to $\mathcal{B}$. In all the cases, z'_{p+1} is a real point different from z'_i, $1 \le i \le p$.

Furthermore, the real rational curves in (2i), (2ii) have no solitary real nodes.

Proof. Assume that $\mathrm{pr}_{\mathcal{B}}(\Delta_0) = 1/2$. Then a generic curve in $|\mathcal{L}(\Delta_0|$ is non-singular and rational. In (1) and (2i) we impose a complete set of generic linear conditions to C_0, which gives the required uniqueness of C_0. In case (2ii), we can define C_0 by an equation

$$F(x,y) := \sqrt{-1}(x\overline{\lambda} - \lambda y)^{2l+1} P(x,y) + Q(x,y) = 0 \ ,$$

where P, Q are given generic homogeneous polynomials, satisfying

$$\overline{P(x,y)} = P(\overline{y}, \overline{x}),\ \overline{Q(x,y)} = Q(\overline{y}, \overline{x}), \quad \deg Q - \deg P = 2l,$$

and $(\lambda, \overline{\lambda}) \in \mathbb{C}P^1 \simeq \mathrm{Tor}(\sigma')$ are unknown coordinates of z'_{p+1}. which can be found from the relation $F(z_1) = 0$ that gives us $2l+1$ solutions.

Assume that Δ_0 is a rectangle with a unit length side parallel to $\mathcal{B}$.

Under conditions (3.41), (3.42), a curve $C_0 \backslash \{z_0\}$ possesses a parameterization $\Pi : \mathbb{C} \to C_0$ such that

$$\Pi(\lambda'_i) = z'_i,\ 1 \le i \le p, \quad \Pi(\lambda''_j) = z''_j,\ 1 \le j \le q, \quad \Pi(\tau) \in \mathrm{Tor}((\Delta_0)_+) \ ,$$

for some conjugation invariant tuples $(\lambda'_1, ..., \lambda'_p), (\lambda''_1, ..., \lambda''_q) \subset \mathbb{C}$ and $\tau \in \mathbb{C} \backslash \mathbb{R}$. We can also assume that a pair of real or conjugate values among $\lambda'_1, ..., \lambda''_q$ are fixed.

So, Π is given by

$$x(t)=\xi\cdot\frac{t-\tau}{t-\overline{\tau}}\cdot\prod_{i=1}^{p}(t-\lambda'_i)^{m'_i}\cdot\left(\prod_{j=1}^{q}(t-\lambda''_j)^{m''_j}\right)^{-1}\ ,$$

$$y(t)=\overline{\xi}\cdot\frac{t-\overline{\tau}}{t-\tau}\cdot\prod_{i=1}^{p}(t-\lambda'_i)^{m'_i}\cdot\left(\prod_{j=1}^{q}(t-\lambda''_j)^{m''_j}\right)^{-1}\ ,$$

where $z_0=(\xi,\overline{\xi})\in(\mathbb{C}^*)^2$. Evaluating these equations at the two fixed parameters among $\lambda'_1,...,\lambda''_q$, we obtain two equations like

$$\left(\frac{a'-\tau}{b'-\overline{\tau}}\right)^2=c',\quad\left(\frac{a''-\tau}{b''-\overline{\tau}}\right)^2=c''$$

with either two real generic triples (a',b',c'), (a'',b'',c''), or two generic complex conjugate triples (a',b',c'), (a'',b'',c''), and in both the cases we obtain four values of τ. Next, for any other parameter λ among $\lambda'_1,...,\lambda''_q$, we obtain an equation like

$$\left(\frac{\lambda-\tau}{\lambda-\overline{\tau}}\right)^2=a$$

with some generic (real or complex) a, which has two solutions. That is we are done in case (2i).

Under conditions (3.43), (3.44), there is a real point among $z'_1,...,z'_p,z''_1,...,z''_q$. Without loss of generality we can assume that this is z''_1. A curve $C_0\backslash\{z''_1\}$ possesses a parameterization $\Pi:\mathbb{C}\to C_0$ such that

$$\Pi(\lambda'_i)=z'_i,\ 1\le i\le p,\quad\Pi(\lambda''_j)=z''_j,\ 2\le j\le q,\quad\Pi(\sqrt{-1})=z_1\ ,$$

with unknown parameters $\lambda'_1,...,\lambda'_p,\lambda''_2,...,\lambda''_q$. We then can write down Π as

$$x(t)=\xi\cdot\frac{t-\tau}{t-\overline{\tau}}\cdot\prod_{i=1}^{p}(t-\lambda'_i)^{m'_i}\cdot(t-\lambda_0)^{2l+1}\cdot\left(\prod_{j=2}^{q}(t-\lambda''_j)^{m''_j}\right)^{-1}\ ,$$

$$y(t)=\overline{\xi}\cdot\frac{t-\overline{\tau}}{t-\tau}\cdot\prod_{i=1}^{p}(t-\lambda'_i)^{m'_i}\cdot(t-\lambda_0)^{2l+1}\cdot\left(\prod_{j=2}^{q}(t-\lambda''_j)^{m''_j}\right)^{-1}\ ,$$

where $z''_1=(\xi,\overline{\xi})\in\mathbb{C}P^1\simeq\mathrm{Tor}(\sigma'')$ and $\Pi(\lambda_0)=z'_{p+1}$. From the relation $\Pi(\sqrt{-1})=z_1$ we derive an equation like

$$\frac{x(\sqrt{-1})}{y(\sqrt{-1})}=\frac{\xi}{\overline{\xi}}\cdot\left(\frac{\sqrt{-1}-\tau}{\sqrt{-1}-\overline{\tau}}\right)^2=a$$

with a given generic $a \in \mathbb{C}^*$ and obtain two values of τ. For each λ'_i, $1 \le i \le p$ (and similarly, for each λ''_j, $2 \le j \le q$), we obtain an equations in the form

$$\frac{x(\lambda'_i)}{y(\lambda'_i)} = \frac{\xi}{\overline{\xi}} \cdot \left(\frac{\lambda'_i - \tau}{\lambda'_i - \overline{\tau}}\right)^2 = a'_i$$

with $a'_i \neq 0$, which gives two values for λ'_i. Finally, we extract the x-coordinate relation from $\Pi(\sqrt{-1}) = z_1$:

$$\xi \cdot \frac{\sqrt{-1} - \tau}{\sqrt{-1} - \overline{\tau}} \cdot \prod_{i=1}^{p} (\sqrt{-1} - \lambda'_i)^{m'_i} \cdot (\sqrt{-1} - \lambda_0)^{2l+1} \cdot \left(\prod_{j=2}^{q} (\sqrt{-1} - \lambda''_j)^{m''_j}\right)^{-1} = a_1 \ ,$$

where a_1 is the first coordinate of z_1, and obtain $2l+1$ (real) solutions for λ_0 (recall that ξ is defined up to a nonzero real factor). Claim (2ii) follows.

To describe the real nodes of C_0 in the case of the rectangular Δ_0, we consider the double cover $C_0 \to \mathbb{C}P^1$ defined by the pencil $\alpha x + \beta y = 0$, $(\alpha, \beta) \in \mathbb{C}P^1$, with two imaginary conjugate ramification points $(1, 0)$ and $(0, 1)$. The above parameterizations show that $\dim C_0(\mathbb{R}) = 1$, and hence C_0 has no solitary real nodes, since otherwise there would be real ramification points. $\Box$

With this statement we finally obtain that the number of admissible tropical limits, corresponding to all odd weights $w(G'_j)$, $j > 2r''_1$, in the combinatorial data, is given by the right hand side of (1.4) if $r'' = 0$, or the absolute value of the right hand side of (1.10), (1.13) with the reduced the term $\prod_{j=1}^{r''_1} (w(G'_{2j}))^2$, if $r'' > 0$.

3.3 Computation of Welschinger invariants

To recover real rational curves $C \in |\mathcal{L}_\Delta|$ on the surface $\Sigma(\mathbb{K})$, passing through $\overline{\boldsymbol{p}}$, we make use of the patchworking theory from [17], section 5, and [18], section 3.

Namely, a tropical limit $(A, \{C_1, ..., C_N\})$ as constructed above, can be completed by deformation patterns, which are associated with the components of the graph G' having weight > 1 and are represented by rational curves with Newton triangles $\mathrm{Conv}\{(0,0), (0,2), (m,1)\}$, m being the weight of the corresponding component of G' (see [17], sections 3.5, 3.6). To obtain a real rational curve $C \in |\mathcal{L}_\Delta|$, we have to choose

- a pair of conjugate deformation patterns for each pair of components of G', corresponding to two conjugate imaginary points of the set Φ (the set of intersection points of the limit curves with the divisors $\mathrm{Tor}(\sigma)$, $\sigma \in E(S)$, $\sigma \perp \mathcal{B}$, as defined in section 2.2);

- a real deformation pattern for each component of G' with weight > 1, corresponding to a real point of Φ.

By [17], Lemma 3.9,

- for any component of G' with weight $m > 1$, there are m (complex) deformation patterns;

- for a component of G' with even weight, corresponding to a real point of Φ, there are no real deformation patterns, or there are two real deformation patterns, one having an odd number of real solitary nodes, and the other having no real solitary nodes,

- for a component of G' with odd weight, corresponding to a real point of Φ, there is precisely one real deformation pattern and it has an even number of real solitary nodes.

For each choice of an admissible tropical limit and suitable set of deformation patterns, Theorem 5 from [17] produces a family of real rational curves $C \in |\mathcal{L}_\Delta|$ on $\Sigma(\mathbb{K})$, which smoothly depends on $r' + 2r'' = |\partial\Delta| - 1$ parameters. Moreover, any curve in the family has the same number of real solitary nodes, equal to the total number of real solitary nodes of the real limit curves and real deformation patterns.

Then we fix the parameters by imposing the condition to pass through $\overline{\boldsymbol{p}}$. By [18], section 3.2, a point $\boldsymbol{p} \in \overline{\boldsymbol{p}}$ such that $\boldsymbol{x} = \mathrm{Val}(\boldsymbol{p})$ is a vertex of A and $\mathrm{Ini}(\boldsymbol{p}) \in (\mathbb{C}^*)^2 \subset \mathrm{Tor}(\Delta_k)$ for some $\Delta_k \in P_{\mathcal{B}}(S)$, gives one smooth relation to the parameters (see [18], formula (3.7)). By [17], section 5.4, a point $\boldsymbol{p} \in \overline{\boldsymbol{p}}$ such that $\boldsymbol{x} = \mathrm{Val}(\boldsymbol{p})$ lies inside an edge of A and $\mathrm{Ini}(\boldsymbol{p}) \in \mathrm{Tor}(\sigma)$ for some $\sigma \in E(S)$, $\sigma \subset \Delta_k$, gives a choice of m smooth equations on the parameters, where m is the intersection number of $\mathrm{Tor}(\sigma)$ with C_k at $\mathrm{Ini}(\boldsymbol{p})$ (see [17], formula (5.4.26)). In the latter case, assume that $\mathrm{Ini}(\boldsymbol{p})$ is real. Then among m relations to parameters there is one real, if m is odd, and there are zero or two real relations if m is even.

Choosing one relation for each point of $\overline{\boldsymbol{p}}$, we obtain a transverse system (see [18], section 3.2), and hence one real rational curve $C \in |\mathcal{L}_\Delta|$ passing through $\overline{\boldsymbol{p}}$.

We notice that whenever an even weight m is assigned to a component of G', corresponding to a real point of the set Φ, then the real rational curves in $|\mathcal{L}_\Delta|$ with a given admissible tropical limit, either do not exist, or can be arranged in pairs with opposite Welschinger numbers due to the choice of two deformation patterns with distinct parity of the number of real solitary nodes (cf. [17], proof of Proposition 6.1), and hence all these real rational curves do not contribute to the Welschinger invariant. If all the weights of the components of G', corresponding to real points

in Φ, are odd, then each admissible tropical limit bears $\prod_{j=1}^{r_1''}(w(G'_{2j}))^2$ real rational curves in $|\mathcal{L}_\Delta|$ due to the choice of $\prod_{j=1}^{r_1''} w(G'_{2j})$ suitable conjugation invariant collections of deformation patterns and the choice of $\prod_{j=1}^{r_1''} w(G'_{2j})$ conjugation invariant collections of relations imposed by the condition $C \supset \overline{\boldsymbol{p}}$. All these curves have the same number of real solitary nodes, and the parity of these numbers coincides with the parity of the total number of the intersection points of the imaginary conjugate components of the limit curves C_k with $\Delta_k \in P_{\mathcal{B}}(S)$. So, the Welschinger numbers of the real rational curves in count are equal to $(-1)^a$ as defined in formulas (1.10), (1.13).

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Address: School of Mathematical Sciences, Tel Aviv University, Ramat Aviv, 69978 Tel Aviv, Israel.

E-mail: shustin@post.tau.ac.il